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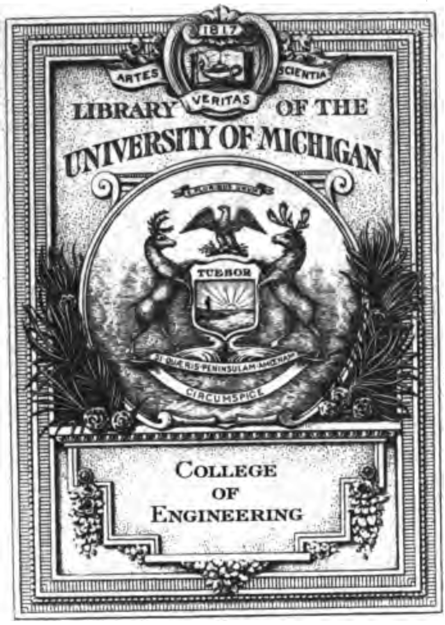

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Carl J. Coe,

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PLANE AND SPHERICAL TRIGONOMETRY

BY

ARTHUR GRAHAM HALL, Ph.D. (Leipzig)
Professor of Mathematics
University of Michigan

AND

FRED GOODRICH FRINK, M.S. (Chicago)
Professor of Railway Engineering
University of Oregon

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PREFACE

This book has had its origin in the desire of the authors to meet the mutual demands of mathematicians and engineers for a treatment that shall more completely supply the needs of the technological student. It is believed that this has been done by enriching the subject with applications to physics and engineering, in such a way as to increase its value at the same time to the general student. The present volume is, moreover, based upon a preliminary edition actually used for several terms in the classroom.

In view of the peculiar situation of trigonometry in the curriculum, the course has been kept of the usual length. The topics have been arranged, however, in the order of increasing difficulty, by postponing the more abstract but no less essential study of the functions of the general angle, until after the arithmetical solution of triangles. The abundance of exercises and problems will give the teacher large opportunity for selection.

The discussion of the slide rule is inserted because of the increasing employment of this useful instrument.

The authors gratefully acknowledge their indebtedness to Professor E. J. Townsend and Professor H. L. Rietz, of the University of Illinois, and Professor A. Ziwet and Professor J. L. Markley, of the University of Michigan, for valuable criticisms and suggestions.

ARTHUR G. HALL.
FRED G. FRINK.

Ann Arbor, January, 1909.

ERRATA

Page 23, Example **3**, for 804.56, read 804.86.

Page 27, **43**, for *sine*, read *arc*.

Page 46, line 11, for ACB, read ACD.

Page 61, first column, last line, for $c = .73367$, read .7337 ; third column, line 3, for 0.51109, read 0.51111 ; third column, last line, for 19.86550 – 20, read 19.86552 – 20.

Page 74, Table, fourth column, for $-\frac{9}{40}$, read $\frac{9}{40}$.

Page 83, § 62, line 2, for $90^\circ \pm a$, read $90^\circ \pm \alpha$; line 3, for $180^\circ \pm a$, read $180^\circ \pm \alpha$.

Page 85, line 4 from bottom, for $= \frac{y'}{x'} = \frac{y}{x} = -\tan\alpha$, read $= \frac{y'}{x'} = -\frac{y}{x} = -\tan\alpha$; last line, for $\frac{v}{x}$, read $\frac{v}{y}$.

Page .96, **14**, for + *sin*, read + cos.

Page 121, last two lines, for $\frac{(1+\tan\gamma)(\cos\gamma-\cot\gamma)}{(1+\cot\gamma)(\sin\gamma-\tan\gamma)} = -\cot\gamma$,

read $\frac{(1+\tan\gamma)(\cos\gamma-\cot\gamma)}{(1+\cot\gamma)(\sin\gamma-\tan\gamma)} = -\cot\gamma\left(\frac{\sin\gamma-1}{\cos\gamma-1}\right)$.

Page 122, **28**, for $\frac{\sin^3\beta+\cos^3\beta}{\sin^2\beta+\cos^2\beta} + \frac{\sin^3\beta-\cos^3\beta}{\sin^2\beta-\cos^2\beta} = -2\sin^2\beta\cos^2\beta$,

read $\frac{\sin^3\beta+\cos^3\beta}{\sin^2\beta+\cos^2\beta} - \frac{\sin^3\beta-\cos^3\beta}{\sin^2\beta-\cos^2\beta} = -\frac{2\sin^2\beta\cos^2\beta}{\sin\beta+\cos\beta}$.

Page 124, **76**, for $2 = \frac{2\tan\alpha}{1+\tan^2\alpha}$, read $2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$; **78**, for 45°, read 90°; **79**, for 45°, read 90°.

Page 129, **168**, line 8, for $CAD = 17^\circ 16'$, read $ADC = 12^\circ 15'$.

ANSWERS

Article **18**, **3**, for 151.5, $c = 381.6$, read 151.2, $c = 335.9$.

Exercise **IX**, **9**, for 524.9, read .64904 ; **6**, for (*b*) .01913, read (*b*) .1913.

Exercise **X**, **1**, for 5.80872, read 6.80872 ; **3**, for 516.35, read 5163.5 ; **15**, for 13.861, read 13.925.

Exercise **XII**, **16**, for $a = 242.79$, $b = 343.16$, read $a = 343.16$, $b = 242.79$.

Article **49**, **1**, for $\gamma = 40^\circ 45.2'$, $a = 450.35$, read $\gamma = 30^\circ 45.2'$, $a = 574.92$.

Exercise **XV**, **6**, for 75.43, read 3017.

Exercise **XVIII**, **8**, for $-\frac{2\sqrt{3}}{3}$, read $\frac{2\sqrt{3}}{3}$; **11**, for $\frac{\sqrt{3}-1}{2}$, read $\frac{1-\sqrt{3}}{2}$.

Exercise **XVIII**, **25**, for $+\sqrt{1-a^2}$, read $+/\sqrt{1}+a^2$.

Exercise **XXI**, **23**, for $2x$, read x.

Exercise **XXII**, **9**, for $\frac{1}{4}$, read $\frac{1}{32}$; **10**, for $\frac{1}{16}$, read $\frac{1}{8}$; **15**, for $k \cdot \frac{\pi}{4}$, read $k \cdot \frac{\pi}{2}$.

Exercise **XXII**, **17**, for $(2k+1)\frac{\pi}{12}$; $[k = 0, 1, 2, \cdots 11]$ and $(3k+1)\frac{\pi}{6}$; $[k = 0, 1, 2, 3]$,

read $(2k+1)\frac{\pi}{12}$; $[k = 0, 1, 2, \cdots 11]$ and $(2k+1)\frac{\pi}{6}$; $[k = 0, 1, 2, 3, 4, 5]$.

Exercise **XXII**, **18**, for $(2k+1)\frac{\pi}{30}$; $[k = 0, 1, 2, \cdots 29]$ and $(2k+1)\frac{\pi}{10}$; $[k = 0, 1, 2, \cdots 9]$,

read $(2k+1)\frac{\pi}{15}$; $[k = 0, 1, 2, \cdots 14]$ and $\frac{2k\pi}{5}$, $[k = 0, 1, 2, \cdots 4]$.

Exercise **XXII**, **19**, $k \cdot \frac{\pi}{4}$; $[k = 0, 1, 2, \cdots 7]$, read $(2k+1)\frac{\pi}{8}$; $(2k+1)\frac{\pi}{4}$.

Exercise **XXIII**, **10**, for $k\frac{\pi}{3}$, read $k\frac{\pi}{3}$; $(6k \pm 1)\frac{\pi}{6}$. **12**, for π and, read $(2k+1)\pi$;

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GREEK ALPHABET

Letters	Names	Letters	Names	Letters	Names
Α α	Alpha	Ι ι	Iota	Ρ ρ	Rho
Β β	Beta	Κ κ	Kappa	Σ σ ς	Sigma
Γ γ	Gamma	Λ λ	Lambda	Τ τ	Tau
Δ δ	Delta	Μ μ	Mu	Υ υ	Upsilon
Ε ε	Epsilon	Ν ν	Nu	Φ φ	Phi
Ζ ζ	Zeta	Ξ ξ	Xi	Χ χ	Chi
Η η	Eta	Ο ο	Omicron	Ψ ψ	Psi
Θ θ	Theta	Π π	Pi	Ω ω	Omega

TRIGONOMETRY

PART I

PLANE TRIGONOMETRY

CHAPTER I

GEOMETRIC NOTIONS

1. General statement. It is assumed that the student is well versed in those theorems of elementary geometry concerning angles, arcs, and triangles. It is particularly desirable that he be familiar with the measurement of angles and with the properties of similar triangles.

While the review thus suggested is left to the student, certain more advanced geometric ideas are treated in the remaining articles of this chapter.

Throughout the course the student should make continual, careful, and intelligent use of such drawing instruments as are included in the equipment at technical schools. In case such sets are not available, as in more general classes, there should be provided at least a straightedge, with graduated scale, a protractor, and a pair of compasses or dividers.

2. Directed line segments. A point which moves from one position to a second, without changing its direction of motion, traces a directed line segment. Directed line segments are always read with regard to their direction, *from* the initial extremity *to* the terminal extremity.

Two line segments are equal if they have the same length and direction, whether their lines are coincident or parallel. Either of two line segments having the same length but opposite directions is said to be the negative of the other. If one direction is taken as positive, the opposite direction is negative.

Thus, in Fig. 1,

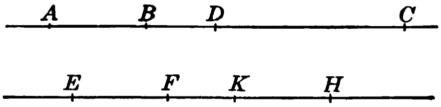

FIG. 1.

$$EF = AB,$$
$$HK = BA = -AB,$$
$$CD = 2\,BA = -2\,AB.$$

If A is the initial point and B the terminal point, the line segment is read AB, and in this notation the positive or negative direction of the segment is expressed without the aid of a prefixed + or −. In case the segment is represented by a single symbol, as the letter a, the direction must be indicated in some further manner, as by a prefixed + or −, or by an arrowhead in the figure.

Two line segments lying in the same line are added by placing the initial point of the second upon the terminal point of the first, each retaining its proper direction. The sum is the segment extending from the initial point of the first to the terminal point of the second. Subtraction is performed by reversing the direction of the subtrahend and adding. Line segments having the same or opposite directions may all be transferred to a common line. Their addition and subtraction thus correspond exactly to the algebraic addition and subtraction of positive and negative numbers.

If A, B, C denote three points arranged in any order along a straight line, then

$$AB + BC = AC,$$

and

$$AB + BC + CA = 0.$$

3. Positive and negative angles. If a line rotates (in a plane) about one of its points, an angle is generated, of which the original position of the line is the initial side and the final position the terminal side. A distinction may be made according as the rotation is clockwise or counter-clockwise about the vertex. The counter-clockwise direction is chosen as positive. Angles are always read with regard to their direction of rotation; thus if OA is the initial side and OB the terminal side, the angle is read AOB. This notation includes the direction or sign of the angle, and the + or − should not be prefixed. In case the angle is represented by a single symbol, as by the Greek letter α, the direction must be indicated in some further manner, as by a prefixed + or −, or by a curved arrow in the figure.

Just as with line segments, reversing the direction multiplies the angle by -1; thus $BOA = -AOB$.

Two angles are added by placing them in the same plane with a common vertex, the initial side of the second coincident with the terminal side of the first, each retaining its own direction. The sum is the angle from the initial side of the first to the terminal side of the second. Subtraction is performed by reversing the subtrahend and adding.

In Fig. 2,

$$AOB + BOC = AOC,$$

$$AOC - BOC = AOB.$$

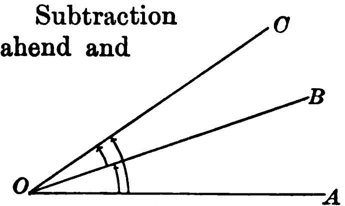

FIG. 2.

EXERCISE I

Solve the following problems graphically:

1. On a train running 40 miles an hour, a man walks 4 miles an hour. Find the speed of the man with reference to the ground, (*a*) if he walks toward the front; (*b*) if he walks toward the rear of the train.

2. The man's speed with reference to the ground is 10 miles an hour. What is the speed of the train (*a*) if he is walking 5 miles an hour toward the front; (*b*) if he is running 8 miles an hour toward the rear?

3. On June 1 the price of corn was 50 cents, and during the succeeding ten days it fluctuated as follows: rose 2 cts., rose 3, fell 1, fell 2, fell 5, fell 3, rose 2, rose 2, rose 3, rose 1. Find the price on June 11.

4. During a football game the progress of the ball from the middle of the field was north 40 yards, south 25, south 5, south 10, south 30, north 50, north 10, north 20. Find the resulting position of the ball.

Combine graphically, using a protractor:

5. $45° + 30°$; $90° + 45°$; $40° + 35° + 50°$.

6. $60° - 45°$; $90° - 50°$; $180° - 120°$.

7. $30° + 80° + 55°$; $40° + 60° - 30°$; $60° - 20° + 70° - 90°$.

8. $40° - 70° + 15°$; $65° + 15° - 90°$; $75° - 180°$.

4. **Rectangular coördinates.** If two mutually perpendicular straight lines are chosen, and a positive direction on each, the position of any point in their plane is determined by giving its perpendicular distances from these fixed lines. The two lines are called the axes of coördinates, and are usually taken so that one

is horizontal and the other vertical. The point of intersection of the axes is called the origin. The two determining data for any point are called its coördinates. The horizontal distance from the axis OY to the point is the abscissa of the point, and the vertical distance from the axis OX to the point is the ordinate of the point. The point whose abscissa is x and ordinate y is denoted by the notation (x, y). Because it is convenient to lay off the abscissa of a point upon the axis OX and the ordinate upon the axis OY, these axes are referred to as the axes of abscissas and ordinates respectively. When x denotes the abscissa and y the ordinate of the point, the axes may be referred to as the X-axis and the Y-axis respectively.

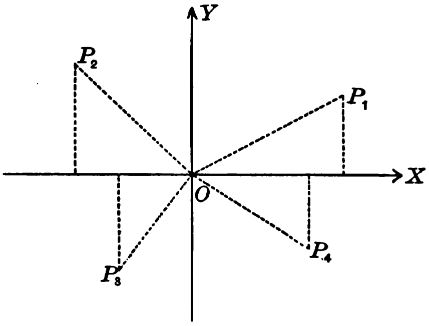

FIG. 3.

The distance from the origin to the point is called the radius vector of the point. It is known whenever the abscissa and the ordinate are given, since the three form, respectively, the hypotenuse, base, and altitude of a right triangle.

The abscissa of a point should always be read *from* the Y-axis *to* the point. The direction from left to right is chosen as positive. Therefore all points at the right of the Y-axis have positive abscissas, and all points at the left, negative abscissas. The ordinate of a point is always read *from* the X-axis *to* the point. The upward direction is chosen as positive. Hence all points above the X-axis have positive ordinates, and all points below, negative ordinates. The radius vector is always read *from* the origin *to* the point, and is always considered positive.

It will be noticed that the abscissa and the ordinate are equal to the projections of the radius vector on the X-axis and Y-axis, respectively; these projections will henceforth be used interchangeably for the coördinates themselves.*

---

* The foot of the perpendicular dropped from a point upon a given line is said to be the orthogonal or orthographic projection of the point on the line. The projection of a line segment on a given line is the segment from the projection of the initial point of the given segment to that of the terminal point. This kind of projection will be used exclusively throughout this book, unless otherwise expressly stated.

The two axes divide the whole plane into four portions, known as the first, second, third, and fourth quadrants, beginning with the upper right-hand quadrant and numbering counter-clockwise about the origin.

If two points, P and Q, lie in a line through the origin, their coördinates, with the radii vectores, form two similar triangles. If the abscissa, ordinate, and radius vector of P are x, y, v, respectively, those of Q are kx, ky, kv.

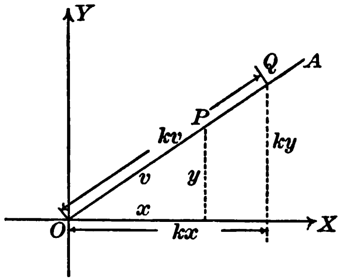

FIG. 4.

EXERCISE II

1. Plot the points (2, 3), (−3, 5), (−2, −4), (1, −3), (3, 0), (0, 4), (−5, 0), (0, −2), (0, 0).

2. Plot the points (3, 2), (6, 4), (12, 8).

3. Plot the points (0, 5), (4, 3), (−3, 4), (0, −5).

4. Find both graphically and by computation the radius vector of each point in examples 1, 2, 3. In what quadrant does each point lie?

5. Describe the location of all points whose abscissas equal 3; whose ordinates equal 5; whose radii vectores equal 6.

6. Write the coördinates of the vertices of a square of side a, if the axes of coördinates are taken along two sides; along its diagonals.

7. The hour hand of a clock is 5 inches long. Find the coördinates of its tip referred to the horizontal and vertical diameters of its face at three o'clock; at six; at eight; at half-past ten.

8. Compare the location of the points (2, 3), (3, 2), (−2, 3), (−2, −3), (3, −2). Describe the directions of their radii vectores.

9. Find the distance from (2, 5) to (6, 9); from (−3, 2) to (4, 5).

10. Find the direction of the line joining (6, 1) to (8, 3); (4, 1) to (1, 4); (6, 3) to (1, 3); (−2, 4) to (1, 1).

11. A man starts from the southwest corner of a government township and goes in turn to the northwest corner of section 36; northwest corner section 22; southeast corner section 3; northeast corner section 5; thence to the point of beginning. Show by sketch the shortest route along section lines, and compute the cross-country distances.

12. A city is laid out in checker-board fashion. The streets are eight to the mile and look to the cardinal points of the compass. It is proposed to introduce two diagonal (45°) streets extending through the busiest corner to the outskirts. What distances will be saved thereby in driving from this corner to each of the points specified below? Nine blocks east and six blocks north; 5 W. and 7 S.; 10 W. and 10 N.; 1 E. and 14 S.

CHAPTER II

THE ACUTE ANGLE

5. Purpose of trigonometry. One of the principal objects of trigonometry is the computation of triangles. We have learned from elementary geometry that a triangle is determined when we know any three of its parts (sides and angles), at least one of them being a side. These data enable us to construct the triangle; but elementary geometry does not teach us how to calculate the remaining parts. The reason is that sides and angles are expressed in different units. It is, therefore, desirable to measure angles not only in degrees, but also by means of lines, or rather by means of ratios of lines. This can be done as shown in the following articles.

6. Definitions of the trigonometric functions. Suppose the acute angle $AOB\ (=\alpha)$ to be placed on a system of axes of coördinates with its vertex at the origin and its initial side OA extending along the X-axis toward the right. Its terminal side OB will fall in the first quadrant. (See Fig. 5.)

Any point P in its terminal side possesses one and only one set of related distances (two coördinates and radius vector). Its abscissa $x\ (=OM)$, its ordinate $y\ (=MP)$, and its radius vector $v\ (=OP)$ are all positive and connected by the relation

$$x^2+y^2=v^2.$$

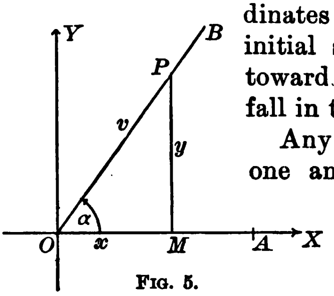

Fig. 5.

The six ratios between these three distances are of frequent occurrence and prime importance. They serve, indeed, the purpose mentioned in Art. 5, and are given the following names and accompanying abbreviations:

$\frac{y}{v}=$ sine of $\alpha = \mathbf{sin}\ \alpha$, $\frac{v}{y}=$ cosecant of $\alpha = \mathbf{csc}\ \alpha$,

$\frac{x}{v}=$ cosine of $\alpha = \mathbf{cos}\ \alpha$, $\frac{v}{x}=$ secant of $\alpha = \mathbf{sec}\ \alpha$.

$\frac{y}{x}=$ tangent of $\alpha = \mathbf{tan}\ \alpha$, $\frac{x}{y}=$ cotangent of $\alpha = \mathbf{cot}\ \alpha$.

7. Relations between the ratios and the angle. The three related distances of any other point P' in the terminal side of the angle α are connected with the determining distances of P by the relation

$$\frac{x'}{x} = \frac{y'}{y} = \frac{v'}{v} = k.$$

It follows that the values of the six ratios defined in Art. 6 do not depend at all upon the particular choice of the point in the terminal side of the angle, but are determined solely and definitely by the size of the angle. A number that is determined in value by the value of some other number is said to be a function of that other number. The six ratios are therefore called *trigonometric functions* of the angle.

This relation between the ratios and the angle is, moreover, a mutual one, such that a knowledge of one of the ratios leads to a knowledge of the angle.* Thus if we have given $\tan \alpha = \frac{2}{3}$, we can construct the angle α as follows: On the system of axes OX and OY plot the point P whose coördinates are $(3, 2)$, using any convenient scale. Draw the line OA from the origin through the point P; then is XOA the required angle α, in consequence of the definitions of Art. 6.

It appears still further that from the knowledge of any one of the six trigonometric functions the remaining five can be found. Thus in the foregoing example we know by the Pythagorean proposition that on the scale employed the hypotenuse or radius vector is $\sqrt{9+4} = \sqrt{13}$. We have then at once

$$\sin \alpha = \frac{2}{\sqrt{13}}, \quad \tan \alpha = \frac{2}{3}, \quad \sec \alpha = \frac{\sqrt{13}}{3},$$

$$\cos \alpha = \frac{3}{\sqrt{13}}, \quad \cot \alpha = \frac{3}{2}, \quad \csc \alpha = \frac{\sqrt{13}}{2}.$$

The properties and relations of these functions and their more immediate applications in pure and applied mathematics constitute the subject-matter of trigonometry. The science takes its name from its origin in the attempts of the ancient peoples to measure triangles.

8. Signs and limitations in value. When any acute angle is placed on the axes of coördinates in the manner prescribed in

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* This statement, as well as some others in the present chapter, will require some modification when we extend the consideration to angles greater than 90°.

Art. 6, its terminal side will always fall in the first quadrant. The abscissa and ordinate, as well as the radius vector, of every point in the terminal side will all be positive. It follows that their ratios, comprising the six trigonometric functions named in Art. 6, are all positive.

In no case can the abscissa or the ordinate of a point be greater than the radius vector. Indeed, save in the exceptional cases considered in Art. 12, they are less than the radius vector. Consequently, $\sin \alpha$ and $\cos \alpha$ cannot be greater than unity and $\sec \alpha$ and $\csc \alpha$ cannot be less than unity.

EXERCISE III

By careful construction and measurement find the approximate values of the following functions:

1. $\cos 60°$. **2.** $\tan 30°$. **3.** $\csc 45°$.

4. $\cot 35°$. **5.** $\sin 20°$. **6.** $\sec 50°$.

Construct each of the following angles and find by measurement the values of all its functions, given

7. $\sin \alpha = \frac{3}{5}$. **8.** $\cos \alpha = \frac{5}{13}$. **9.** $\tan \alpha = \frac{7}{24}$.

10. $\cot \alpha = \frac{8}{15}$. **11.** $\sec \alpha = \frac{41}{9}$. **12.** $\csc \alpha = \frac{61}{11}$.

13. $\cos \alpha = \frac{12}{37}$. **14.** $\sin \alpha = \frac{21}{29}$.

15. For what angle is the tangent equal to the cotangent? For what angle is the sine equal to the cosine?

16. Show that the direct functions ($\sin \alpha$, $\tan \alpha$, $\sec \alpha$) are greater or less than the corresponding complementary functions ($\cos \alpha$, $\cot \alpha$, $\csc \alpha$) respectively, according as the angle α is greater or less than 45°.

17. Can $\sin \alpha$ and $\tan \alpha$ be equal? When do they approach equality?

18. Show that $\tan \alpha \cdot \cot \alpha$ does not depend on α. Show that the same is true of $\sin \alpha \cdot \csc \alpha$.

19. Show that $\cos \alpha \cdot \sec \alpha$ does not depend on α. Show the same for $\csc^2 \alpha - \cot^2 \alpha$.

20. Does $\sin^2 \alpha + \cos^2 \alpha$ depend on α? Does $\sec^2 \alpha - \tan^2 \alpha$?

21. Before reading Art. 11, find the values of the sine, cosine, and tangent of 30°, 45°, and 60°.

9. Fundamental relations. The Pythagorean theorem

$$x^2 + y^2 = v^2,$$

and the definitions of Art. 6 yield certain fundamental relations between the six trigonometric functions of a single angle. Thus, we have directly from the definitions

$$\csc \alpha = \frac{1}{\sin \alpha}, \tag{1}$$

$$\sec \alpha = \frac{1}{\cos \alpha}, \tag{2}$$

$$\cot \alpha = \frac{1}{\tan \alpha}. \tag{3}$$

Again, by division,

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}, \tag{4}$$

and

$$\cot \alpha = \frac{\cos \alpha}{\sin \alpha}. \tag{5}$$

Dividing by v^2 both members of the equation

$$y^2 + x^2 = v^2,$$

we have

$$\left(\frac{y}{v}\right)^2 + \left(\frac{x}{v}\right)^2 = 1,$$

whence

$$\sin^2 \alpha + \cos^2 \alpha = 1. \tag{6}$$

Dividing, in like manner, by x^2 and by y^2 respectively, we obtain

$$\tan^2 \alpha + 1 = \sec^2 \alpha, \tag{7}$$

$$\cot^2 \alpha + 1 = \csc^2 \alpha. \tag{8}$$

These eight equations constitute the fundamental relations of trigonometry. Of these the fifth, seventh, and eighth may be derived algebraically from the other five. By means of these equations it is possible to express all six functions in terms of any one of them. If the value of any one is given, the values of the others can be found. Simple numerical examples of the latter kind are more quickly solved by the geometrical method of Art. 7.

EXAMPLE 1. To find the remaining functions of the acute angle whose tangent is $\frac{5}{12}$.

(1) *Geometric Method.* The right triangle *OMP* (Art. 6, Fig. 5) has sides of relative length $y = 5$, $x = 12$, whence on the same scale $v = 13$. Thus the sine is $\frac{5}{13}$, etc.

(2) *Analytic Method.* Given $\tan\alpha = \frac{5}{12}$. Then by the formulas of Art. 9,

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{12}{5};\quad \sec\alpha = \sqrt{1+\tan^2\alpha} = \frac{13}{12};$$

$$\csc\alpha = \sqrt{1+\cot^2\alpha} = \frac{13}{5};$$

$$\cos\alpha = \frac{1}{\sec\alpha} = \frac{12}{13};\quad \sin\alpha = \frac{1}{\csc\alpha} = \frac{5}{13}.$$

Example 2. To express all the functions of α in terms of $\sec\alpha$. By the use of the appropriate formulas of Art. 9, we obtain

$$\cos\alpha = \frac{1}{\sec\alpha};\quad \sin\alpha = \sqrt{1-\cos^2\alpha} = \frac{\sqrt{\sec^2\alpha - 1}}{\sec\alpha};$$

$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{\sec\alpha}{\sqrt{\sec^2\alpha - 1}};\quad \tan\alpha = \sqrt{\sec^2\alpha - 1};$$

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{1}{\sqrt{\sec^2\alpha - 1}}.$$

Example 3. Verify the following relation by reducing the first member to the second:

$$\frac{\tan^2\beta}{\sec\beta - 1} - 1 = \sec\beta.$$

By means of the formulas of Art. 9, we have

$$\frac{\tan^2\beta}{\sec\beta - 1} - 1 = \frac{\sec^2\beta - 1}{\sec\beta - 1} - 1 = \sec\beta + 1 - 1 = \sec\beta.$$

EXERCISE IV

By means of the formulas of Art. 9, find the values of the remaining functions of each of the following angles, given

1. $\sin\alpha = \frac{17}{145}$. **2.** $\cot\beta = \frac{56}{33}$. **3.** $\cos\gamma = \frac{28}{53}$. **4.** $\tan\gamma = \frac{19}{180}$.

Express all the functions of α in terms of

5. $\tan\alpha$. **6.** $\cos\alpha$. **7.** $\cot\alpha$. **8.** $\sin\alpha$.

Find the values of the following expressions:

9. $\dfrac{\tan\alpha + \cot\alpha}{\tan\alpha - \cot\alpha}$, when $\cos\alpha = \dfrac{9}{41}$.

10. $\dfrac{\sec\alpha - \cos\alpha}{\tan\alpha - \sin\alpha}$, when $\sin\alpha = \dfrac{12}{13}$.

11. $\frac{\sin\beta}{1+\cos\beta}+\cot\beta$, when $\tan\beta=\frac{20}{21}$.

12. $\cos\beta\cdot\tan\beta+\sin\beta\cdot\cot\beta$, when $\sec\beta=\frac{65}{33}$.

Express the following in terms of a single function:

13. $\frac{\csc\alpha}{\cot\alpha+\tan\alpha}$ in terms of $\cos\alpha$.

14. $\frac{\sin\beta}{1+\cos\beta}+\frac{1+\cos\beta}{\sin\beta}$ in terms of $\csc\beta$.

15. $\sec\gamma-\tan\gamma$ in terms of $\sin\gamma$.

16. $\frac{1}{1-\sin\gamma}+\frac{1}{1+\sin\gamma}$ in terms of $\tan\gamma$.

Verify the following identities:

17. $\cos^4\beta-\sin^4\beta=2\cos^2\beta-1$.

18. $\cos\alpha\cdot\tan\alpha=\sin\alpha$.

19. $\frac{\cot^2\alpha}{1+\cot^2\alpha}=\cos^2\alpha$.

20. $\frac{1}{\tan^2\beta+1}+\frac{1}{\cot^2\beta+1}=1$.

10. Functions of complementary angles. If, in Fig. 6, $\angle XOB$ is constructed equal to $\angle AOY$, XOB $(=\beta)$ and XOA $(=\alpha)$ are complementary. The triangles $OM'P'$ and OMP are similar, the pairs of corresponding sides being v' and v, x' and y, y' and x.

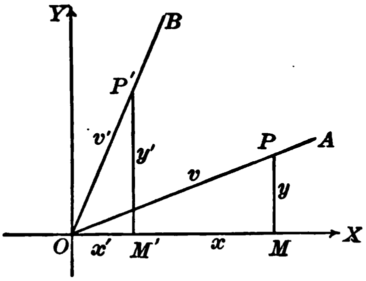

FIG. 6.

We have then

$$\sin(90^\circ-\alpha)=\sin\beta=\frac{y'}{v'}=\frac{x}{v}=\cos\alpha,$$

$$\cos(90^\circ-\alpha)=\cos\beta=\frac{x'}{v'}=\frac{y}{v}=\sin\alpha,$$

$$\tan(90^\circ-\alpha)=\tan\beta=\frac{y'}{x'}=\frac{x}{y}=\cot\alpha,$$

$$\cot(90^\circ-\alpha)=\cot\beta=\frac{x'}{y'}=\frac{y}{x}=\tan\alpha,$$

$$\sec(90^\circ - \alpha) = \sec\beta = \frac{v'}{x'} = \frac{v}{y} = \csc\alpha,$$

$$\csc(90^\circ - \alpha) = \csc\beta = \frac{v'}{y'} = \frac{v}{x} = \sec\alpha.$$

The prefix "co" is thus seen to be the abbreviation of the word "complement's." The general theorem may be stated as follows:

Any trigonometric function of an acute angle is equal to the corresponding co-function of its complementary angle.

Thus, $\sin 72^\circ = \cos 18^\circ$, $\cot 54^\circ = \tan 36^\circ$, etc.

11. Functions of 45°, 30°, 60°. The exact values of the functions of certain angles are readily found.

(1) If, in Fig. 7, $\angle XOA = 45^\circ$, the triangle OMP is isosceles, so that $x = y = \frac{1}{\sqrt{2}}v$. We have at once

$$\sin 45^\circ = \cos 45^\circ = \tfrac{1}{2}\sqrt{2},$$

$$\tan 45^\circ = \cot 45^\circ = 1,$$

$$\sec 45^\circ = \csc 45^\circ = \sqrt{2}.$$

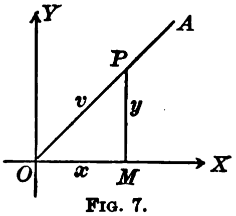

FIG. 7.

(2) If, in Fig. 8, $\angle XOA = 30^\circ$, and $\angle XOC$ is constructed equal to -30°, the triangle QOP is equilateral, so that $y = \frac{1}{2}v$, $x = \frac{1}{2}\sqrt{3}\,v$.

We have, at once,

$$\sin 30^\circ = \tfrac{1}{2}, \quad \cos 30^\circ = \tfrac{1}{2}\sqrt{3},$$

$$\tan 30^\circ = \tfrac{1}{3}\sqrt{3}, \quad \cot 30^\circ = \sqrt{3},$$

$$\sec 30^\circ = \tfrac{2}{3}\sqrt{3}, \quad \csc 30^\circ = 2.$$

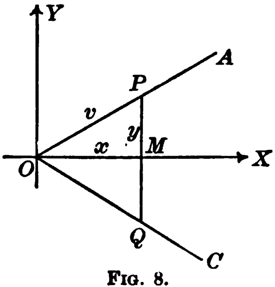

FIG. 8.

(3) In like manner to (2), or by Art. 10, we obtain

$$\sin 60^\circ = \tfrac{1}{2}\sqrt{3}, \quad \cos 60^\circ = \tfrac{1}{2},$$

$$\tan 60^\circ = \sqrt{3}, \quad \cot 60^\circ = \tfrac{1}{3}\sqrt{3},$$

$$\sec 60^\circ = 2, \quad \csc 60^\circ = \tfrac{2}{3}\sqrt{3}.$$

12. Functions of 0° and 90°.

(1) As the $\angle XOA$ (see Fig. 9) decreases so as to approach the limit zero, the abscissa x will approach equality to the radius

vector v. If, at the same time, the radius vector remains finite, the ordinate y will approach the limit zero.

It should be noticed that the cosecant varies in such a manner that its denominator approaches the limit zero while its numerator remains constantly equal to the finite number v, so that the value of the cosecant increases without limit as the angle approaches zero. It is then said to become infinite and is represented by the symbol ∞. The cotangent also becomes infinite as the angle approaches zero, since its numerator, although changing, never exceeds v.

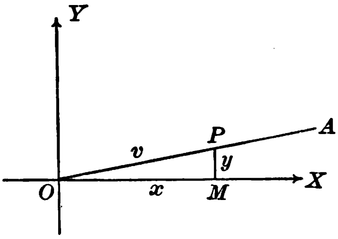

FIG. 9.

We have, then,

$$\sin 0^\circ = 0, \quad \cos 0^\circ = 1,$$
$$\tan 0^\circ = 0, \quad \cot 0^\circ = \infty,$$
$$\sec 0^\circ = 1, \quad \csc 0^\circ = \infty.$$

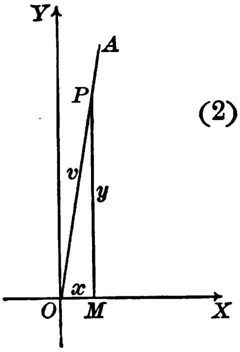

FIG. 10.

(2) In like manner, we obtain

$$\sin 90^\circ = 1, \quad \cos 90^\circ = 0,$$
$$\tan 90^\circ = \infty, \quad \cot 90^\circ = 0,$$
$$\sec 90^\circ = \infty, \quad \csc 90^\circ = 1.$$

EXAMPLE. Solve the equation $\sec^2\gamma - \sqrt{3}\tan\gamma = 1$.

Expressing wholly in terms of $\tan\gamma$,

we have $\tan^2\gamma + 1 - \sqrt{3}\tan\gamma - 1 = 0$,

or $\tan^2\gamma - \sqrt{3}\tan\gamma = 0$,

whence $\tan\gamma = 0$ and $\sqrt{3}$.

Then, by Arts. 11 and 12,

$$\gamma = 0^\circ \text{ and } 60^\circ.$$

EXERCISE V

1. Express in terms of an angle less than 45° the functions of 75°.
2. Express in terms of an angle less than 45° the functions of 65°.

Verify the following for $\alpha = 30°$; also for $\alpha = 45°$:

3. $\sin 2\alpha = 2 \sin \alpha \cos \alpha$.

4. $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$.

5. $\tan 2\alpha = \dfrac{2 \tan \alpha}{1 - \tan^2 \alpha}$.

6. $\cot 2\alpha = \dfrac{\cot^2 \alpha - 1}{2 \cot \alpha}$.

Notice that $\sin 2\alpha$ does *not* equal $2 \sin \alpha$, $\cos 3\alpha$ does *not* equal $3 \cos \alpha$, etc.

Verify for $\alpha = 30°$:

7. $\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$.

8. $\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$.

Evaluate the following expressions:

9. $\sin 60° \cos 30° - \cos 60° \sin 30°$.

10. $\cos 60° \cos 30° + \sin 60° \sin 30°$.

11. $\csc^2 45° + \sin 60° \tan 30°$.

12. $\sin 60° \tan 45° - \sec 30° \sin^2 45° \cot 60°$.

Solve each of the following equations for some one function of α and find the angle in degrees. Verify the results by substitution in the given equation.

13. $\tan \alpha - 3 \cot \alpha = 0$.

14. $\sec \alpha + 2 \cos \alpha = 3$.

15. $4 \sin^2 \alpha + 3 \cot^2 \alpha = 4$.

16. $\sin \alpha + 3 \cos \alpha = 2\sqrt{2}$.

17. A ladder 22 feet long leans against a wall, its foot being 11 feet away from the wall. Find (*a*) the angle formed by the ladder with the ground; (*b*) the height of the top of the ladder above the ground.

18. The diagonal of a rectangle makes an angle of 30° with the long side. If the length of this side is 14 inches, what is the length of the short side of the rectangle? of the diagonal?

19. The side of a conical pile of sand makes an angle of 45° with the floor. If the height from the floor is 12 feet, what is the area of the base?

20. A guy rope (assumed to be straight) has a length of 60 feet and extends from the top of a mast 30 feet high to the ground. Find the angle between the rope and the mast.

13. Variation of the trigonometric functions as the angle varies. Suppose the angle θ to vary continuously from 0° to 90°. The terminal side revolves about the origin from the position *OX* to the position *OY*. If v remains constant, y will increase from 0 to v, while x will decrease from v to 0. Consequently, the numerator of the fraction $\frac{y}{v}$ $(= \sin \theta)$ increases from 0 to v, while the denominator remains constant. Hence, while θ increases from 0° to 90°, $\sin \theta$ increases from 0 to 1.

The numerator of the fraction $\frac{x}{v}$ $(= \cos \theta)$ decreases from v to 0, while the denominator remains constantly equal to v. Hence, while θ increases from 0° to 90°, $\cos \theta$ decreases from 1 to 0.

The numerator of the fraction $\frac{y}{x}$ $(= \tan \theta)$ increases from 0 to v, while the denominator decreases from v to 0. Hence, while θ increases from 0° to 90°, $\tan \theta$, beginning with zero, increases without limit as θ approaches 90°. We express this by saying that $\tan \theta$ varies from 0 to ∞.

The student should trace carefully the variation of the other trigonometric functions and compare the results with the values found in Arts. 11 and 12. Article 7 should be read again at this point.

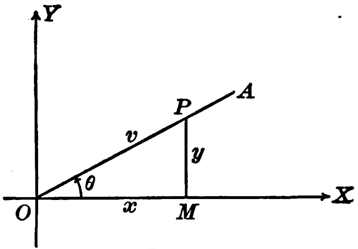

FIG. 11.

14. Inverse trigonometric functions. The same functional relation is expressed by the two statements, "m is the sine of the acute angle α" and "α is the acute angle whose sine is m." The corresponding symbolic notations are

$$m = \sin \alpha, \qquad \alpha = \text{arcsin}\, m,^*$$

with the understanding that α is an acute angle and that m is a positive number not greater than unity. A similar symbolic relation holds for the other trigonometric functions. It is frequently read "arc-sine m," or "anti-sine m," since two mutually inverse functions are said each to be the anti-function of the other.

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* This notation is universally used in Europe and is fast gaining ground in this country. A less desirable symbol,

$$\alpha = \sin^{-1} m,$$

is still found in English and American texts.

The notation $\alpha = \text{inv}\sin m$ is perhaps better still on account of its general applicability. (See Art. 80.)

The inverse notation is convenient for the statement of problems. The purposes of interpretation and manipulation are better served by transforming to the corresponding direct notation.

EXAMPLE. Find the value of $\sin(90^\circ - \operatorname{arccot} \frac{5}{12})$.

In the direct notation the example reads: Given $\cot \alpha = \frac{5}{12}$, find $\sin(90^\circ - \alpha)$. Then, by Arts. 10 and 9,

$$\sin(90^\circ - \alpha) = \cos \alpha = \tfrac{5}{13}.$$

EXERCISE VI

1. Trace the variation in value of $\sec \theta$.
2. Trace the variation in value of $\csc \theta$.

Find the values of the following:

3. $\tan(\cos^{-1} \frac{8}{17})$.
4. $\sin(\operatorname{arccot} \frac{4}{3})$.
5. $\cos(90^\circ - \arctan \frac{7}{24})$.
6. $\cot(90^\circ - \sin^{-1} \frac{16}{65})$.
7. $\sec(90^\circ - \operatorname{arcsec} 2)$.
8. $\csc(90^\circ - \operatorname{arccsc} \sqrt{2})$.
9. $\sin(2 \tan^{-1} 1)$.
10. $\cos(2 \sin^{-1} \frac{1}{2})$.

Solve the following equations:

11. $2 \sin^2 \beta + 3 \cos \beta - 3 = 0$.
12. $\sec \beta - 2 \tan \beta = 0$.
13. $\tan \beta (2 \sin \beta - 1)(\sec \beta - \sqrt{2}) = 0$.
14. $\sin \beta (2 \cos \beta - \sqrt{3})(\tan \beta - 1) = 0$.

Verify the identities:

15. $\sin^4 \alpha + \cos^4 \alpha + \sin^2 \alpha \cos^2 \alpha = \sin^4 \alpha - \sin^2 \alpha + 1$.
16. $(\csc \alpha - \cot \alpha)(\csc \alpha + \cot \alpha) = 1$.
17. $(\tan \alpha + \cot \alpha)(\sin \alpha \cdot \cos \alpha) = 1$.
18. $1 - \tan^4 \alpha = 2 \sec^2 \alpha - \sec^4 \alpha$.
19. $\sin^6 \alpha + \cos^6 \alpha = 1 - 3 \sin^2 \alpha \cos^2 \alpha$.
20. $\cos^4 \alpha - \sin^4 \alpha = 1 - 2 \sin^2 \alpha$.

15. Orthogonal projection. In accordance with the definitions of Art. 4 (see note, page 4) it follows that the projection of a line segment on any line is equal to the length of the line segment multiplied by the cosine of the angle formed by the line segment with the line of projection. Thus, in Fig. 12, the projection of AB on RS is

$$MN = AB \cos \alpha.$$

In like manner, the projection of AB on a line perpendicular to RS (*i.e.* making $+90°$ with RS) has the value $AB \sin \alpha$. These projections are called the *components* of the line segment AB along and at right angles to the direction RS.

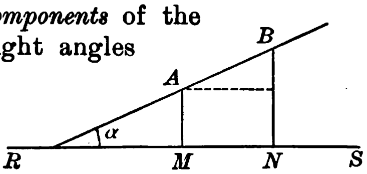

FIG. 12.

In physics, line segments are often used to represent quantities that have direction as well as magnitude; for example, forces, velocities, accelerations. The components of the line segment used to represent a force represent components of the force; likewise for a line segment representing a velocity, acceleration, or moment. Suppose, for example, that the line segment AB, Fig. 13, represents a force applied to the block m resting on a horizontal plane. This segment has the component EB parallel to the plane and the component FB perpendicular to the plane. Segment EB represents a force component F_x parallel to the plane, which tends to move the block along the plane; segment FB represents a force component F_y perpendicular to the plane and tending to produce pressure between the block and plane. Denoting by F the force represented by AB, we have

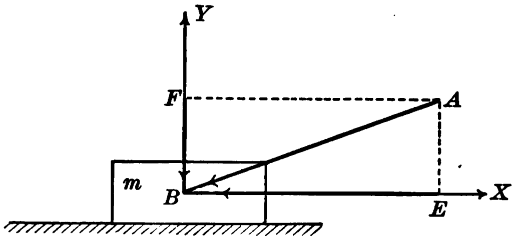

FIG. 13.

$$F_x = F \cos \alpha, \qquad F_y = F \sin \alpha.$$

EXAMPLE. At a given instant a point is moving in a direction at an angle of 30° with a given horizontal line with a velocity of 20 feet per second. Find the component of the velocity along the line.

Taking the given line as the X-axis, we have for the component v_x

$$v_x = v \cos 30° = 20 \times \tfrac{1}{2}\sqrt{3} = 17.321 \text{ feet per second.}$$

The component along a line perpendicular to the given horizontal line in the plane of motion is

$$v_y = v \sin 30° = 20 \times \tfrac{1}{2} = 10 \text{ feet per second.}$$

EXERCISE VII

The student should draw appropriate figures for each of the following exercises.

1. Find the projections of a line segment 8.5 inches in length on the X- and Y-axes, (*a*) when the segment makes an angle of 45° with the X-axis; (*b*) when it makes an angle of 60° with the Y-axis.

2. A crank 16 inches long rotates in a vertical plane. When the crank makes an angle of 30° with the horizontal diameter of the circle described by the moving end, what is the distance of the moving end, (*a*) from the horizontal diameter? (*b*) from the vertical diameter?

3. If in Fig. 13 the force F denoted by AB is 40 pounds, find the components F_x and F_y, (*a*) when $\alpha = 30°$; (*b*) when $\alpha = 45°$. Discuss the cases $\alpha = 0°$ and $\alpha = 90°$.

4. A steamer is moving at a speed of 18 miles per hour in a direction north of east, making an angle of 30° with an east and west line. At what rate is the steamer sailing eastward? at what rate northward?

5. A guy wire exerts a pull of 3000 pounds on its anchorage and makes an angle of 30° with the ground. Find the component of this force, (*a*) along the ground; (*b*) vertical.

6. The eastward and northward components of the velocity of a moving body are found to be $v_x = 12$ miles per hour and $v_y = 12\sqrt{3}$ miles per hour, respectively. Find (*a*) the magnitude and (*b*) the direction of the body's velocity.

CHAPTER III

RIGHT TRIANGLES

16. Laws for solution. If, in a right triangle, two independent parts are known, in addition to the right angle, the three remaining parts can be found. Thus two given parts, at least one of which is a side, determine a right triangle. The formulas needed in all cases to effect this solution are five in number. Two are the statements of well-known geometric theorems, while the other three are the immediate consequences of the definitions of the trigonometric ratios contained in Art. 6.

In Fig. 14 let ACB be a right triangle, right-angled at C. We shall denote the interior angles at the vertices by α, β, γ, and the lengths of the sides opposite them by a, b, c, respectively. Note that $\gamma = 90°$, and c is the hypotenuse.

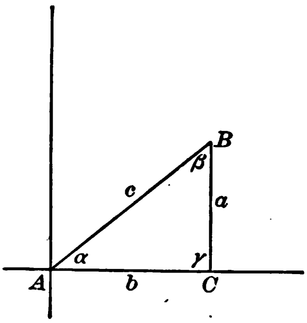

FIG. 14.

The five formulas are the following:

$$a^2 + b^2 = c^2, \tag{1}$$

$$\alpha + \beta = 90°, \tag{2}$$

$$\frac{a}{c} = \sin\alpha = \cos\beta, \tag{3}$$

$$\frac{b}{c} = \cos\alpha = \sin\beta, \tag{4}$$

$$\frac{a}{b} = \tan\alpha = \cot\beta. \tag{5}$$

Equation (1) follows from the Pythagorean theorem, and (2) from the fact that the sum of the angles of a triangle is equal to two right angles. In order to establish the last three, place the triangle on the axes of coördinates described in Art. 4, the side AC extending from the origin to the right along the X-axis, and the hypotenuse lying in the first quadrant, as in Fig. 14.

Then b, a, c, are respectively the abscissa, ordinate, and radius vector of B, a point on the terminal side of the angle α, which is conventionally placed.

It follows at once from Art. 6 that

$$\frac{a}{c} = \sin \alpha,$$

$$\frac{b}{c} = \cos \alpha,$$

$$\frac{a}{b} = \tan \alpha.$$

The corresponding values of the functions of the angle β result from Art. 10.

17. Area of right triangles. The formulas for the area of a right triangle follow from the familiar geometric theorem

$$\text{Area} = \tfrac{1}{2} \times \text{base} \times \text{altitude},$$

or expressed symbolically,

$$A = \tfrac{1}{2} ab. \tag{1}$$

The substitution for a of its value from the preceding article gives

$$A = \tfrac{1}{2} bc \sin \alpha. \tag{2}$$

Again, introducing the values of both a and b,

$$A = \tfrac{1}{2} c^2 \sin \alpha \cdot \cos \alpha. \tag{2}$$

Other formulas for the area may also be obtained.

18. Method of solution. The solution of any problem consists of four parts: the analysis, the algebraic solution, the arithmetical computation, and the interpretation of the results.

(1) The student should read and analyze the problem, noting which parts are known and which are desired. The construction of a neat and sufficiently accurate figure is helpful and advisable.

(2) The student should select from the five formulas of Art. 16 those containing a single unknown part each, in addition to the known parts, and should solve them for these unknown parts while still in the literal form.

Experience has led to the adoption of the following two rules of procedure:

(*A*) The use of the Pythagorean formula, $a^2 + b^2 = c^2$, is to be avoided save when the data are very simple or a table of squares and square roots is at hand.

(*B*) So far as is consistent with rule *A*, each unknown part should be found in terms of those parts originally given in the problem, in order to avoid accumulation of errors.

In conformity with these rules, the angle relation $\alpha + \beta = 90°$, and two of the three trigonometric formulas serve to effect the solution, while the remaining trigonometric formula affords a check on the work.

(3) The solution is now effected by introducing the numerical data and performing the necessary computations. The correctness and accuracy of the results are greatly enhanced by extreme orderliness of arrangement and neatness of detail.

The use of the trigonometric tables and the employment of suitable checks will be discussed in subsequent articles.

(4) The geometric or physical significance of the results obtained should be fully considered and interpreted.

EXAMPLE 1. Given $c = 254$, $\alpha = 30°$, to find a, b, β.

In this instance the analysis and construction are obvious.

The three appropriate formulas yield at once the forms

$$\beta = 90° - \alpha,$$

$$a = c \sin \alpha,$$

$$b = c \cos \alpha.$$

The formula $b = a \tan \beta$ affords the check.

On introducing the numerical data, we obtain

$$\beta = 90° - 30° = 60°,$$

$$a = 254 \times \tfrac{1}{2} = 254 \times .5 = 127,$$

$$b = 254 \times \tfrac{1}{2}\sqrt{3} = 254 \times .86605 = 219.9767.$$

The check formula gives

$$b = 127 \times 1.7321 = 219.9767.$$

EXAMPLE 2. Given $a = 39.00$, $b = 80.00$, to find c, α, β, and the area.

As before, we may pass immediately to the second stage. Now c is given directly in terms of a and b by the formula $c = \sqrt{a^2 + b^2}$.

If we are to avoid the use of this formula, we must first find α and β, and then get c by means of one of these angles. We use the forms:

$$\tan\alpha = \frac{a}{b},$$

$$\beta = 90^\circ - \alpha,$$

$$c = \frac{a}{\sin\alpha},$$

$$A = \tfrac{1}{2}ab,$$

and for the check $$c = \frac{b}{\sin\beta}.$$

We obtain, then $$\tan\alpha = 39 \div 80 = .4875,$$

$$\alpha = 25^\circ\,59', \text{ as found from Table III},$$

$$\beta = 90^\circ - 25^\circ\,59' = 64^\circ\,01',$$

$$c = 39 \div .4381 = 89.01,$$

$$A = \tfrac{1}{2} \times 39 \times 80 = 1560,$$

and for the check $$c = 80 \div .8989 = 89.00,$$

showing a difference of .01.

On account of the simplicity of the numbers, we may, by using the formula $c^2 = a^2 + b^2$, find, exactly, $c = 89$.

Explain the accumulation of errors and, hence, the reason for rule of procedure (*B*).

EXAMPLES. 1. Given $c = 42$, $\beta = \arcsin .28$; find a and b.

2. Given $b = 27$, $\alpha = \tan^{-1} .75$; find a and c.

3. Given $a = 300$, $\alpha = \cos^{-1} .45$; find c and b.

4. Given $c = 200$, $\alpha = \operatorname{arccot} 1.12$; find a and b.

19. Trigonometric tables. In the first example worked in the preceding article, the functions of 30° had been determined in Art. 11. In the second example, however, the value of $\tan\alpha$ was not one of those previously ascertained, and the value of α was not recognizable from its tangent. For convenience of reference the numerical values of the sines, cosines, tangents, and cotangents of all angles differing by intervals of one minute from 0° to 90° have been collected in Table III, on pages 71–89. The arrangement is simple and plain. The degree numbers from 0° to 44° occur at the top of the page, with the minutes running down the left margin.

The numerical values of the functions, computed to four decimal places, are placed in columns under the names of the functions.

Since $\sin(90° - \alpha) = \cos\alpha$, and $\tan(90° - \alpha) = \cot\alpha$, the space required may be reduced one half. The degree numbers of angles from 45° to 90° are printed at the bottom of the pages in reversed order, the minutes run up the right margin, and the names of the functions are in reversed order at the bottom.

For the present the student need not concern himself with smaller divisions of the angle than the minute. Further refinement is attained by a method to be described in Art. 26.

Table IV, on pages 91–93, contains the squares of numbers less than 1000 and, by interpolation, of numbers up to 9999. The first page gives directly the squares of numbers from 1 to 100. On the second and third pages the tens and units digits of the number to be squared are in the left margin, while the hundreds digits are at the tops of the several columns. The last two figures of the square are in the column at the right under U., opposite the tens and units digits; the first three, or four, figures of the square are in the same line in the column under the hundreds digit. In the right margin are the last two figures of the tabular difference used in interpolation, to which must be prefixed the remainder obtained by subtracting the first three, or four, figures of the square from those in the same column immediately beneath, or that remainder diminished by 1 when the asterisk (*) is present. The use of the table is best shown by illustration.

EXAMPLES. 1. $328^2 = 107{,}584$.

2. $475.3 = 475^2 + .3 \times 951 = 225{,}625 + 285 = 225{,}910$.*

3. $28.37^2 = 28.3^2 + .07 \times 567 = 800.89 + 3.97 = 804.86$.

Square roots are extracted by reversing the process; thus,

4. $\sqrt{27556} = 166$.

5. $\sqrt{658{,}037} = \sqrt{657{,}721} + 316 \div 1623 = 811 + .2 = 811.2$.

20. Errors and checks. The results obtained are not always, nor even usually, exactly correct. The deviations from the true values are of two sorts, mistakes and errors, and a sharp distinction must be made between them.

---

* This result is, of course, only approximately correct. The true result may be obtained as follows:

$$475.3^2 = 475^2 + .3 \times (475.3 + 475) = 225{,}625 + 285.09 = 225{,}910.09.$$

The data for problems arising in actual practice are derived from observations made with instruments for measuring lengths, angles, etc.

Mistakes may arise from a false reading of the observing instrument, a misapprehension of the problem, the employment of the wrong formula, faulty addition, etc. They are never allowable or excusable.

On the other hand, instruments are so constructed as to yield results only to a certain degree of precision, which should be ascertained for each instrument. Moreover, observation is performed by the human apparatus, eyes, ears, etc., and a certain personal equation, an anticipation or lagging in sight or hearing, is always present, varying with personal fitness and experience. Methods of eliminating instrumental errors, so as to obtain the maximum precision possible with the instruments used, are given in standard works on engineering instruments. Again, the arithmetical calculation involves the trigonometric ratios, which are, in general, non-terminating decimal fractions, while their values in the mathematical tables are computed only to a certain number of decimal places. Errors, therefore, will always be present; but every precaution should be taken to keep the errors due to computation well within the limits of error of the observed data and desired results fixed by the nature of the problem.

In both observation and solution, certain additional processes are employed to avoid, or to reveal, mistakes. These processes are known as checks and vary with the nature of the problem.

While no general rules for checks can be laid down, a frequent practice in the solution of triangles is to make use of a formula connecting the required parts, just found, noting if the results are within the range of allowable error. The size of this allowable error should be known for each table.

As a check to arithmetical computation, graphical construction is well understood and strongly advised. As a means of avoiding the grosser mistakes, a free-hand sketch will frequently suffice by guiding the student to a reasonable interpretation of data, and indicating possible results.

A drawing constructed to scale will further aid by yielding values more or less approximate, approaching those obtained by computation.

Carried a step farther as regards accuracy, by the use of precise instruments, the graphical construction often attains to the dignity of an independent solution, with results falling within the limits prescribed by the physical conditions of the problem.

There is no better evidence of careful work than the recording of a reasonable error obtained by the comparison of two methods. In practical work the allowable per cent of error becomes an important consideration.

EXERCISE VIII

Find the missing parts of the following triangles, using the natural trigonometric functions, Table III.

	α	β	a	b	c	A
1.	25° 10′				34	
2.	52° 20′				73	
3.		61° 15′			243	
4.		78° 35′			521	
5.	21° 25′		235			
6.	72° 45′		720			
7.		80° 30′	1200			
8.		17° 30′		1500		
9.			240		360	
10.			381		715	
11.				521	630	
12.				840	1400	
13.			648	864		
14.			595	600		
15.			215	385		
16.			2111	1234		
17.			95			7980
18.				264		30360
19.	74° 20′					1225
20.		24° 50′				843

21. In the same vertical plane the distances shown in Fig. 15 were measured in feet along the surface of the ground. The distances of the different points below the instrument, as measured by a rod, are given also in feet. The vertical scale is exaggerated for clearness. What is the horizontal distance from B to G? (Check by a table of squares and square roots.)

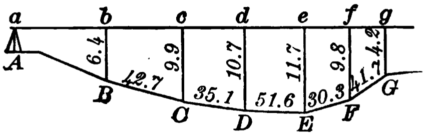

FIG. 15.

22. A line surveyed across a ridge is 1500 feet in horizontal length. Stakes are set 100 feet apart horizontally by level chaining. By leveling, the elevations of the surface at the different stakes is obtained as follows: 730.2, 735.9, 739.7, 743.4, 750.1, 751.8, 760.7, 764.1, 764.3, 765.8, 765.0, 763.2, 758.3, 750.2, 743.1, 740.2. What length of wire will be required for fencing along this line? (Check by a table of squares and square roots.)

23. If a gravel roof slopes one half inch to the horizontal foot, what angle does it make with the horizon?

24. If the face of a wall has a batter or inclination of one inch in one vertical foot, what is its angle with the vertical?

25. What is the angle of ascent of a railway built on a 2 per cent grade (*i.e.* 2 vertical feet to 100 horizontal feet)?

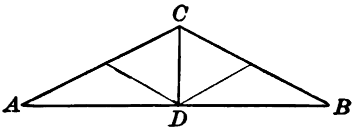

FIG. 16.

26. The pitch of a roof is the ratio $\frac{CD}{AB}$. (See Fig. 16.) What is the inclination to the horizon of a roof with $\frac{1}{3}$ pitch, $\frac{1}{4}$ pitch, $\frac{1}{2}$ pitch?

27. What is the pitch of a roof sloping to the horizon at 15°, 30°, 45°?

28. What is the inclination to the horizon of the corner or hip rafter of a pyramidal roof whose pitches are $\frac{1}{4}$?

29. What is the inclination from the vertical of the corner edge of a wall, both of its faces having a batter of $\frac{1}{10}$?

30. At what angle does a railway slope if it has a grade of 0.25%, 0.5%, 2.5%?

31. At what angle must a cog railway ascend in order to rise 2640 feet in one horizontal mile?

32. A battleship known to be 341 feet long subtends an angle of 3° 20′ when presenting its broadside to a fort on shore. For what distance should guns be sighted when trained upon it? (Note that the isosceles triangle having the length of the ship for its base is separable into two right triangles.)

33. In planning the stairway for a house it is desired that the riser, or vertical distance between steps, shall be 7 inches, and the treads, or horizontal distances between faces, 11 inches. What will be the angle of inclination of the hand rail?

34. Taking the data of the preceding problem, what will be the length of the hand rail if straight, provided the height between floors is 11 feet 8 inches?

35. A cylindrical water tower whose external diameter is 25 feet subtends a horizontal angle of 5° 30′ as viewed from a distance. How far is its center from the instrument?

(Note that we have a triangle that is right-angled when the line of sight is tangent. The base is the radius of the tower and the opposite angle is half of the one observed.)

36. What horizontal angle would be subtended, at a distance of 2 miles, by a vertical cylindrical gas receiver 60 feet in diameter?

(See note to problem 35.)

37. The end of a pendulum 34 inches long swings through an arc of $3\frac{1}{4}$ inches. Find the angle through which the pendulum swings.

38. When vertically over a village, a balloon's angle of inclination, as viewed from 9 miles distant, was 15° 20′. Assuming the surface of the country to be fairly level, what was the height of the balloon?

39. A flagstaff 110 feet high is covered by a vertical angle of 12° 30′ at a point approximately on a level with its center. How far is the observer from the staff?

40. The data of a preliminary survey are as follows:

$AB = 240.9$ feet. Angle at $B = 62° 11'$ left.
$BC = 310.7$ feet. Angle at $C = 55° 50'$ left.
$CD = 611.5$ feet. Angle at $D = 43° 42'$ right.
$DE = 237.2$ feet. Angle at $E = 51° 23'$ right.
$EF = 528.0$ feet.

Considering A, Fig. 17, as the origin of coördinates and AB as the axis of abscissas, it is required to compute coördinates for all points given, thus providing for the accurate mapping of the survey.

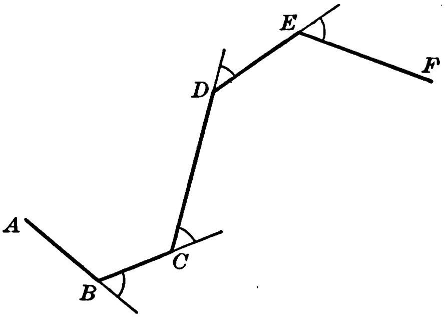

FIG. 17.

41. Find the missing parts and area of the following isosceles triangles (see Fig. 18 for lettering):

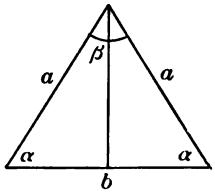

FIG. 18.

$\alpha = 35°$, $a = 42$; $\alpha = 72°$, $b = 125$;
$a = 350$, $b = 180$; $\beta = 54°$, $a = 360$;
$\beta = 51° 26'$, $b = 480$; $a = 640$, $b = 840$.

42. Find the lengths of the chords of the following arcs in terms of the radius: 30°, 36°, 40°, 45°, 60°, 75°, 90°, 120°. Compute, given $R = 100$.

43. Express in terms of the ~~sine~~ arc and radius the relation between the chord of an arc and the chord of half the arc.

44. Express in trigonometric form the most important relations between the radius R of the circumscribed circle, the radius r of the inscribed circle, the side s, and the number of sides n of a regular polygon.

45. Compute and tabulate the perimeter and the circumferences of the circum- and in-circles of a regular polygon of n sides for $n = 4, 8, 16, 32$, given $R = 10$.

46. Compute and tabulate the area of a regular polygon of n sides and of its circum- and in-circles for $n = 4, 8, 16, 32$, given $R = 10$.

47. Repeat example 45 for $n = 6, 12, 24, 48$.

48. Repeat example 46 for $n = 6, 12, 24, 48$.

49. A body is acted upon by three forces of magnitudes 20, 40, 60, parallel to the sides of an equilateral triangle. Resolve these forces along two perpendicular axes, then combine, and thus find the magnitude and direction of the resultant.

50. A body situated at one vertex of a regular hexagon is acted upon by five forces represented in magnitude and direction by the vectors drawn to the five other vertices. Resolve along and perpendicular to the diameter through the point and find the magnitude and direction of the resultant.

51. A point describes a circle with uniform speed. Determine the position of its projection upon a diameter in terms of its angular displacement from that diameter.

52. A point describes a circle of radius 30 feet at a rate of 4 revolutions per minute. Find the position of its projection upon a diameter at the end of 5 seconds after passing the extremity of that diameter.

53. Determine the components of the vertical acceleration g along and perpendicular to a plane inclined at an angle α to the horizon.

54. If $g = 32$, find the acceleration along and perpendicular to a plane whose inclination to the horizontal is 30°, 15°, 10°, 5°.

55. A man weighing 150 pounds stands midway on a 30-foot ladder whose foot is 10 feet horizontally from the vertical wall against which it leans. Find the normal (perpendicular) pressure on the ladder and the force tending to cause him to slide along the ladder.

56. Find the components along the X- and Y-axes of a force of 65 pounds making an angle of $28° 13'$ with the X-axis.

57. A steamer is sailing in such a way that its speed due east is 12 miles per hour and its speed due south is 14 miles per hour. Find the direction of the steamer's course and the speed in that course.

58. In an oblique triangle, angle $B = 45°$, angle $C = 32°$, and side $b = 16$. Find side c. (SUGGESTION. Draw the perpendicular from the vertex A upon the opposite side.) Attempt to deduce a general relation between the functions of the acute angles of an oblique triangle and the opposite sides.

CHAPTER IV

LOGARITHMS

21. Definition of a logarithm. If we have given

$$56 = 10^{1.74819},\ 79 = 10^{1.89763},$$

we can find the product of 56 and 79 without performing the operation of multiplication, provided we know in advance the powers of 10. For, we have from the general laws governing exponents,

$$\begin{aligned} 56 \times 79 &= 10^{1.74819} \times 10^{1.89763} \\ &= 10^{1.74819+1.89763} \\ &= 10^{3.64582} = 4424. \end{aligned}$$

It will be seen that the process of multiplication has been replaced by the simpler one of addition.

Many other processes in computation can be simplified in a similar manner; for example, if we wish to find the cube root of a number, say 89.1, we have

$$89.1 \doteq 10^{1.94988},$$

and consequently

$$\sqrt[3]{89.1} = (10^{1.94988})^{\frac{1}{3}} = 10^{0.64996} = 4.466^{+}.$$

In this case the extraction of a root has been accomplished by the simple process of division. In order to extend this method we must know all of the powers of some convenient number. The exponents involved are called logarithms, and the number raised to a power is referred to as the base of the logarithmic system. We may define a logarithm more exactly as follows:

If a is any number and x and n are so related that $a^x = n$, then x is called the *logarithm* of n to the base a; that is, a logarithm is the index of the power to which the base must be raised to obtain the given number.

This relation is denoted symbolically by writing

$$x = \log_a n,$$

and is read "x is equal to the logarithm of n to the base a."

Thus 3 is the logarithm of 8 to the base 2, since $2^3 = 8$; and in the illustrations given above, 1.74819 is the logarithm of 56 to the base 10, etc.

The two statements

$$a^x = n, \quad x = \log_a n$$

are inverse to each other, just as are the relations sin x and arcsin x, etc., of Art. 14.

Exercise. Find by inspection $\log_3 27$, $\log_5 .625$, $\log_8 32$, $\log_5 .04$.

The logarithm of a number to itself as base is unity, since $n^1 = n$.

The logarithm of 1 to any base other than zero is zero, since $a^0 = 1$.

In conformity with the definition just laid down, it follows that, if two numbers are equal, their logarithms to the same base are equal. It is also true conversely, that if the logarithms of two numbers to the same base are equal, the numbers are equal.*

If the base is real and positive, real logarithms produce only positive numbers. If the base is real and negative, even logarithms produce positive numbers; odd logarithms, negative numbers. For this reason only real positive bases are chosen in practice, and only positive numbers are combined by the aid of their logarithms. The sign of the result is ascertained entirely apart from the numerical computation.

22. Laws of combination. Logarithms are important in trigonometry and elsewhere as labor-saving devices in calculations with numbers containing many digits. Only so much of the theory of logarithms as is necessary for this purpose will be developed in the present chapter.

The laws of combination of numbers by the aid of their logarithms follow at once from the definition of the preceding article.

I. *The logarithm of the product of two factors is equal to the sum of their logarithms, all to the same base.*

For, if $x = \log_a n$ and $y = \log_a m$ we may write

$$n = a^x \text{ and } m = a^y.$$

---

* In the theory of analytic functions a broader definition of the logarithm is laid down, and the statement just made requires modification.

Multiplying, we have, by the exponential law,

$$nm = a^{x+y},$$

whence, $$\log_a nm = x + y = \log_a n + \log_a m. \tag{1}$$

This law may evidently be extended to any finite number of factors.

II. *The logarithm of the quotient is equal to the logarithm of the dividend minus the logarithm of the divisor, all to the same base.*

For, if $x = \log_a n$ and $y = \log_a m$, we may write as before,

$$n = a^x, \quad m = a^y.$$

Dividing, we have $\frac{n}{m} = a^{x-y}$,

whence, $$\log_a \left(\frac{n}{m}\right) = x - y = \log_a n - \log_a m. \tag{2}$$

Manifestly $\log_a \left(\frac{1}{m}\right) = -\log_a m.$

III. *The logarithm of the power of a number is equal to the logarithm of the number multiplied by the index of the power.*

For, if $x = \log_a n$, then $n = a^x$.

Hence, $$n^p = (a^x)^p = a^{px}$$

or, $$\log_a (n^p) = px = p \log_a n. \tag{3}$$

IV. *The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.*

For, if $x = \log_a n$, then $n = a^x$. Extracting the qth root of both members, we get

$$\sqrt[q]{n} = a^{\frac{x}{q}},$$

whence, $$\log_a \sqrt[q]{n} = \frac{x}{q} = \frac{1}{q} \log_a n. \tag{4}$$

23. Common logarithms. Any number may be used as a base of a system of logarithms. For certain purposes the so-called natural system of logarithms, which has for its base the number $e = 2.71828183 \cdots$, has advantages. For the purposes of ordinary numerical computation, however, it is most convenient to employ for the base of the system of logarithms, 10, the base of the universally adopted system of numeration.

The common logarithms of all exact integral powers of 10 are positive integers; for instance

$$\begin{aligned}\log_{10}(1000000) &= \log_{10}(10^6)\\ &= 6\log_{10}10\\ &= 6.\end{aligned}$$

The logarithms of reciprocals of integral powers of 10 are negative integers; thus

$$\begin{aligned}\log_{10}(.00001) &= \log_{10}(10^{-5})\\ &= -5\log_{10}10\\ &= -5.\end{aligned}$$

The logarithms of numbers situated between two consecutive integral powers of 10, say between 10^k and 10^{k+1}, lie between k and $k+1$, where k is any integer, positive or negative. Thus

$$10^3 < 2417 < 10^4,$$

whence, $$3 < \log_{10} 2417 < 4,$$

or, $$\log_{10} 2417 = 3 + \text{a number lying between 0 and 1.}$$

The logarithms of numbers greater than the base consist of an integer plus a proper fraction. The fractional part is written decimally, calculated to a number of decimal places, depending on the degree of accuracy desired in the use of the table. The integral part of the logarithm is called the *characteristic;* the decimal fraction, its *mantissa.*

Hereafter, in this book, except in Chapter IX, we shall have to do only with common logarithms and, unless otherwise expressly stated, $\log n$ will denote $\log_{10} n$.

24. Characteristic. If one number is equal to another number multiplied by a factor which is a power of 10, the logarithms of the two numbers differ by an integer. For

$$\begin{aligned}\log(10^k \times n) &= \log(10^k) + \log n\\ &= k + \log n.\end{aligned}$$

EXAMPLE. $$\begin{aligned}\log 34000 &= 3 + \log 34\\ &= 4 + \log 3.4, \text{ etc.}\end{aligned}$$

Every number containing one digit at the left of the decimal point lies between 10^0 and 10^1. The characteristic of its logarithm is therefore 0. The cipher should always be written to indicate that the characteristic has not been overlooked.

Every number containing k digits at the left of the decimal point is 10^{k-1} times a number with one digit at the left. The characteristic is therefore $k-1$. We have then the following rule for the characteristic:

The characteristic of the logarithm of any number greater than unity is one less than the number of digits at the left of the decimal point.

Should the number be less than unity, move the decimal point ten places to the right (thus multiplying by 10^{10}) and apply the same rule as before, then write -10 after the logarithm for correction. Thus

$$\begin{aligned}\log 7.12 &= 0.85248,\\ \log 71200 &= \log(10^4 \times 7.12)\\ &= 4.85248,\\ \log .00712 &= \log(10^{-10} \times 71200000)\\ &= \log(10^{-10} \times 10^7 \times 7.12)\\ &= 7.85248 - 10.\end{aligned}$$

The positive part of the last characteristic is seen to be the difference found by subtracting from 9 the number of ciphers immediately following the decimal point in the number.

The characteristic of the logarithm of any number less than unity is found by subtracting from 9 *the number of ciphers between the decimal point and the first significant digit, then affixing* -10.

25. Mantissa. We have seen that moving the decimal point in the number merely changes the characteristic of the logarithm, leaving its mantissa unaltered. The mantissa depends solely upon the sequence of significant digits.

In the tables given, the logarithms are computed to five decimal places (see pp. 1–21), and the mantissas alone for all numbers from 100 to 9999 are given, arranged in the following manner: Running down the left margin, under N, are to be found the first three digits of the number. In the next (open) column occur the first two figures of the mantissa. In the next ten columns are the remaining three figures of the mantissa arranged under the fourth digit of the number at the top of the columns.

Thus to find the mantissa of log 3814, we select the row having 381 in the left margin. The first two figures of the mantissa, 58, are found in the first column. The three remaining figures, 138, are found in the column headed 4, the fourth digit of the number, giving the mantissa .58138.

To avoid repetition, the first two figures, 58, are not printed in every line, but are to be used from 3802 to 3890, inclusive. The prefixed asterisk, *006, denotes that the mantissa of 3891 is .59006, not .58006.

EXERCISE IX

1. Find by inspection $\log_2 16$, $\log_9 27$, $\log_4 \frac{1}{32}$.

2. Find by inspection $\log_3 81$, $\log_8 32$, $\log_{27} 9$.

3. What numbers correspond to the following logarithms to base 4: 0, 1, 2, 2.5, 3, -2, -3?

4. What numbers correspond to the following logarithms to base 8: 0, 1, $1\frac{1}{3}$, $-\frac{2}{3}$, -2?

5. Find by logarithms: (*a*) $\frac{693}{1467}$; (*b*) $\frac{.0872 \times 144}{778}$.

6. Find (*a*) $\sqrt{793}$; (*b*) $\sqrt[3]{.007}$; (*c*) $\sqrt[5]{91}$.

7. Find $\sqrt{\dfrac{18 \times \sqrt{240} \times 75^3}{72 \times \sqrt{640} \times 200}}$.

8. Find $\left(\dfrac{32 \times \sqrt[3]{720} \times 15^4}{72 \times \sqrt{480} \times 24^3}\right)^{\frac{2}{3}}$.

9. Find $\left(\dfrac{14.7}{65}\right)^{\frac{k-1}{k}}$, where $k = 1.41$.

10. Solve for x: $4^x = 24$.

11. Solve for x: $6^x = 25$.

The amount A attained by a principal P at interest at the rate r compounded annually for n years is

$$A = P(1 + r)^n.$$

12. Find the amount of \$3680 at 4 per cent in 6 years.

13. Find the principal which, in 7 years at 5 per cent, amounts to \$5820.

14. At what rate will \$5000 amount to \$7500 in 8 years?

15. In how many years will \$86,500 amount to \$129,600 at $3\frac{1}{2}$ per cent?

16. If a city increases its population $\frac{1}{5}$ each year, in how many years will it double its size?

26. Interpolation. It will be shown in Art. 79 that the difference in the logarithms of two numbers is approximately proportional to the difference in the numbers provided these differences are small. Thus, approximately,

$$\frac{\log 51473 - \log 51470}{\log 51480 - \log 51470} = \frac{51473 - 51470}{51480 - 51470} = \frac{3}{10}.$$

We have, then,

$$\log 51473 = \log 51470 + \tfrac{3}{10}(\log 51480 - \log 51470).$$

Introducing the values from Table I,

$$\begin{aligned}\log 51473 &= 4.71155 + \tfrac{3}{10}(4.71164 - 4.71155)\\ &= 4.71155 + .3 \times .00009\\ &= 4.71155 + .00003\\ &= 4.71158.\end{aligned}$$

The difference .00009, or omitting the denominator, the 9 is called the tabular difference corresponding to the logarithm of 5147. Note that the added difference is computed to the nearest fifth decimal place.

This process is called interpolation by the principle of proportional parts. To facilitate interpolation, tables of proportional parts are inserted in the logarithmic tables in the column headed P.P. At the top of each of the P.P. tables is the tabular difference and under this is the number to be added corresponding to the digit at the left. For example

$$\log 38.25 = 1.58263$$

$$\log 38.26 = 1.58274.$$

The difference is .00011 and in the P.P. column is a table headed 11. Suppose now that log 38.257 is required. Opposite 7 under 11 is found 7.7 ; hence 8 is to be added in the fifth decimal place, giving

$$\log 38.257 = 1.58271.$$

27. Numbers from given logarithms. The inverse process of finding the number corresponding to a given logarithm is best explained by an illustration. Given the logarithm 3.84235. Only the mantissa need be considered at first, as the characteristic merely determines the position of the decimal point in the number.

Looking for 84 in the first column after the margin, we find it corresponding to numbers from 692 to 707. The nearest tabular number (mantissa) smaller than 235 is 230, corresponding to the number 6955. The difference is 5, while the tabular difference, found by subtracting 230 from 236, is 6. We have now the proportion for the next digit,

$$\frac{n}{10}=\frac{5}{6};$$

so that the next digit is found by dividing 50 by 6. It is inadvisable to carry the interpolation beyond one additional digit. Since $50 \div 6 = 8 \cdot + \cdots$, we have found the desired number to be 6955.8. The decimal point is placed after the fourth digit according to the rule for the characteristic, the given characteristic being 3. Should the logarithm be followed by -10, the decimal point must finally be moved ten places to the left.

28. Cologarithms. The logarithms of divisors have to be subtracted. Subtraction, however, can be avoided and the logarithmic computation of a succession of multiplications and divisions effected by a single addition process. There is no advantage in using cologarithms when but two factors are involved. When, however, more than two are involved, instead of dividing by the denominator or divisor factors, we may multiply by their reciprocals, obviously a legitimate substitution. Now

$$\log\frac{1}{m} = -\log m = (10-\log m)-10.$$

This logarithm, $(10-\log m)-10$, is called the cologarithm of m, written $\operatorname{colog} m$. It may be written down immediately from the table by beginning at the left and subtracting each figure from 9, until the last figure, which must be subtracted from 10. Thus

$$\log 28.24 = 1.45086$$

and

$$\operatorname{colog} 28.24 = 8.54914 - 10.$$

29. Logarithms of trigonometric functions. Logarithms of the trigonometric functions are arranged in Table II in the same manner as are the natural functions, or true numerical values of the functions. Logarithmic secants and cosecants need not be printed, since they are the cologarithms of the cosines and sines.

The sines and cosines of angles and the tangents of angles less than 45° are numerically less than unity. In conformity with Art. 24, therefore, their logarithms are written in the augmented form,

$$\log \sin 65^\circ\, 21' = 9.95850 - 10.$$

The -10 is not printed in the table but it is always understood. The positive portion of the characteristic is printed in the table. Usage differs with respect to printing the logarithmic tangents of angles greater than 45°. Engineering and physical instruments are usually graduated to minutes or larger divisions of the angle, so that it is not feasible to carry the interpolation farther than to tenths of minutes. The tables of functions and of proportional parts printed in connection with this book are arranged with this in view.

Astronomic observations justify carrying the interpolation to seconds, and astronomers use for this purpose tables computed to seven or more decimal places.

For example,

$$\log \sin 29^\circ\, 37' = 9.69390 - 10,$$

$$\log \sin 29^\circ\, 38' = 9.69412 - 10.$$

The difference is .00022, and in the P.P. column is a table headed 22. Suppose now that log sin 29° 37.4′ is required. Opposite 4 under 22 is found 8.8; hence 9 is to be added in the fifth decimal place, giving

$$\log \sin 39^\circ\, 37.4' = 9.69399 - 10.$$

EXERCISE X

1. Find from the table the logarithms of 72484, 619.25, 695×10^7, .00064375, 3×10^{21}.

2. Find from the table the logarithms of 91386, 14.295, 321×10^9, .000078541, 2×10^{24}.

3. Find the numbers whose logarithms are 3.71295, 12.61242, 8.21312 – 10.

4. Find the numbers whose logarithms are 4.21382, 11.75153, 6.13579 – 10.

5. Find Young's modulus of elasticity from the formula $Y = \frac{mgl}{\pi r^2 s}$, if $m = 4932.5$, $g = 980$, $l = 110.5$, $\pi = 3.1416$, $r = .25$, $s = .3$.

6. Find the radius of the sun if its mass is 2.03×10^{33} grams, and its average density is 1.41, knowing that mass = volume × density.

7. The radius r of each of two equal, tangent, iron spheres which attract each other with a force of 1 gram's weight, is given by the formula

$$\frac{(\frac{4}{3}\pi \rho r^3)^2}{4r^2} = \frac{M}{R^2},$$

in which the density of iron $\rho = 7.5$, the mass of the earth $M = 6.14 \times 10^{27}$ grams, and the radius of the earth $R = 6.37 \times 10^{8}$ cm., while $\pi = 3.1416$. Solve for r and compute by logarithms.

8. Solve example 7 for spheres of lead with density $\rho = 11.3$.

9. The population of a county increases each year by 12.5 per cent of the number at the beginning of the year. If its population Jan. 1, 1776, was 2.5×10^{6}, what will it be Dec. 31, 1926?

10. If the number of births and deaths per annum are 3.5 per cent and 1.2 per cent respectively of the population at the beginning of each year, and the population on Jan. 1, 1830, was 5×10^{5}, find the population Jan. 1, 1905.

11. Find from the tables log sin 25° 32.3′, log cot 71° 18.6′, colog cos 16° 29.2′.

12. Find from the tables log cos 19° 25.7′, log tan 31° 16.2′, colog sin 65° 12.8′.

13. Find the angles corresponding to $\log \cos \alpha = 9.31723$, $\log \cot \beta = 9.16251$, $\log \tan \gamma = 0.61253$.

14. Find the angles corresponding to $\log \sin \alpha = 9.63152$, $\log \tan \beta = 9.71728$, $\log \cot \gamma = 0.15382$.

15. Francis deduces the following formula for the discharge over a weir, $q = 3.01\, bH^{1.53}$, in which q is the discharge in cubic feet per second, b the breadth of the crest, and H the head of water. Find by logarithms the discharge when $b = 3.5$ and $H = 1.2$.

16. A common formula for finding the diameter of a water pipe is

$$d = 0.479 \left[\frac{flq^2}{h}\right]^{\frac{1}{5}}$$

in which f is a friction factor, l the length of the pipe, q the discharge, and h the head. Find d when $f = 0.02$, $l = 500$, $q = 5$, $h = 10$.

17. The discharge from a triangular weir is given as $q = c\,\frac{8}{15}\sqrt{2\,g}\,H^{\frac{5}{2}}$, in which c is a constant, g the acceleration of gravity, and H the head. Find q when $g = 32.2$, $H = 1.2$, $c = 0.592$.

18. The formula for velocity head is $h = 0.01555\, V^2$. Find h when $V = 5$.

19. The elevation of the outer rail on what is known as a one-degree railway curve to resist centrifugal force is sometimes given by the formula $e = 0.00066\, V^2$, e being in inches and V the speed of the train in miles per hour. When $V = 45$, compute e.

20. Another expression for the relation of the preceding problem is $e = \dfrac{gV^2}{32.2\,R}$. Here e is in feet, g is the gauge of the track, V is the speed in feet per second, and R is the radius of the curve. Given $g = 4.71$, $V = 66$, $R = 5730$, compute e.

21. The difference between the base and hypotenuse of a right triangle is given by $c - a = \dfrac{b^2}{c+a}$, and when a and c are nearly equal, approximately by $c - a = \dfrac{b^2}{2c}$.

Find the per cent of error introduced by the second method when the angle between a and c is 15°.

22. If a = length of a short circular arc and c = its chord, then approximately $a - c = \dfrac{a^3}{24R^2}$. Given $a = \dfrac{2\pi R}{24}$ and $R = 100$, compute the value of this difference.

23. The relation between the pressure and volume of air expanding under certain conditions is $p_1 v_1{}^{1.41} = pv^{1.41}$, where p_1 and v_1 are initial values. If $p_1 = 40$, $v_1 = 5.5$, find v when $p = 24$; also when $p = 16$.

24. The relation between the initial and final temperatures and pressures is given by the equation

$$\frac{t + 460}{t_1 + 460} = \left(\frac{p}{p_1}\right)^{\frac{.41}{1.41}}.$$

With $t_1 = 60$ and the other data as in Ex. 23, find the final temperatures for $p = 24$ and $p = 16$, respectively.

30. The slide rule. The principles of logarithmic computation are conveniently illustrated by means of the slide rule, now widely used in performing mechanically such operations as admit of the use of logarithms. A brief description of this instrument will be found profitable at this stage, and its use by the student as a ready check upon the numerical solution of problems is strongly recommended. As will be seen by an inspection of the simplified diagram of Fig. 19, the rule is essentially a device for adding and subtracting logarithms, thereby giving a wide range of computations. In the figure the point a on the "slide" is set opposite the point B on the "rule." If both scales, which are alike, are so divided that AB is equal, or proportional, to log 2 and ab to log 3, then C on the rule opposite b on the slide gives the distance AC equal, or proportional, to log 6. That is, log 2 + log 3 = log (2 × 3) = log 6.

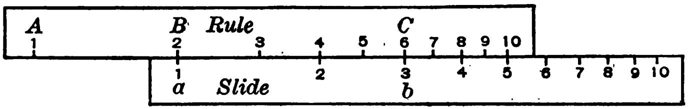

FIG. 19.

Similarly by subtraction, $AC - ab = AB$;

that is $\log 6 - \log 3 = \log 2.$

The point *a* of the slide is called the *index*, hence we have the following rules for simple operations.

1. To multiply two numbers, set the index opposite one number on the rule and opposite the other number on the slide read the product on the rule.

2. To divide one number by another, set the divisor on the slide opposite the dividend on the rule and read the quotient on the rule opposite the index.

In the instrument as actually constructed, * Fig. 20, there are four scales denoted respectively by A, B, C, and D, of which scales B and C are on the slide. For convenience in compound operations the rule is provided with a runner r by means of which a setting of the slide may be preserved while the slide is moved to a new position. The following example will illustrate the manipulation of slide and runner.

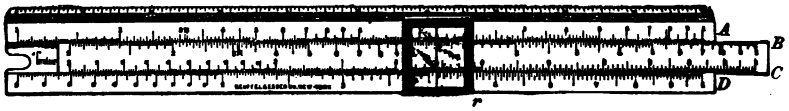

FIG. 20.

EXAMPLE 1. Find $\frac{63 \times 115 \times 27}{14.6 \times 342}$.

Set 14.6 on C scale opposite 63 on D scale; move runner to 115 on C scale; move 342 on C scale to runner, and opposite 27 on C scale read result on D scale.

In this, as in all slide-rule computations, the decimal point must be located by inspection.

On the lower side of the slide are three scales, the outer of which are marked S and T respectively. The following examples illustrate the use of these scales.

EXAMPLE 2. Find 36 sin 22°.

Set 22 on the S scale opposite the mark on the slot in the right-end of the rule; then opposite the end of the A scale can be read on the B scale the natural sine of 22°. Now opposite 36 on the A scale read the result on the B scale.

EXAMPLE 3. Find 26.5 tan 13° 15′.

Reverse the slide and set 13° 15′ on the T scale opposite the mark on the slot; then opposite the end of the B scale can be read on the D scale the natural tangent of 13° 15′. Set the runner at this point and replace the slide with the index at the runner. Opposite 26.5 on the C scale read the required product on the D scale.

EXAMPLE 4. Find $56^{1.8}$.

Set the index of C scale opposite 56 on D scale and opposite the mark on the under side of the right-hand end of the rule read 748 on the middle scale of the lower side of the slide. This reading is the mantissa of the logarithm of 56. The characteristic 1 must be supplied as usual. Now in the usual way find 1.3 × 1.748; that is, put index to 1.748 on *D* scale and opposite 1.3 on *C* scale read the product 2.272. This is the logarithm of $56^{1.3}$. Set the mantissa 272 on the logarithm scale opposite the mark on the rule and read 118.7 on the *D* scale opposite the index.

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* A more detailed description of the slide rule is not within the scope of this book. A manual describing fully the use of the instrument can be had of any firm selling slide rules.

EXERCISE XI

1. Find (*a*) $\frac{64 \times 37}{163}$; (*b*) $\frac{193}{67 \times 2.1}$; (*c*) $\frac{0.05 \times 137 \times 62}{14 \times 28 \times 6.5}$.

2. Find (*a*) 127 sin 24°, (*b*) 0.32 sin 72°, (*c*) 16.5 cos 35°.

3. Find (*a*) 37 tan 8° 20′, (*b*) 1.35 tan 40° 10′.

4. Find (*a*) $\frac{17 \cot 32^\circ}{64}$, (*b*) $35.5 \frac{\sin 32^\circ}{\sin 47^\circ}$.

5. Find (*a*) $28^{\frac{1}{4}}$, (*b*) $\sqrt[3]{3.65}$, (*c*) $7.31^{1.27}$.

31. Right triangles solved by logarithms. — It is now possible, with the aid of the logarithmic tables, to solve right triangles the numerical values of whose parts contain more digits than those given in Chapter III, without entailing laborious multiplications and divisions.

EXAMPLE 1. Given $a = 51.72$, $\beta = 73^\circ\ 46'$.

Solving the proper formulas for the unknown parts, we have

$$\alpha = 90^\circ - \beta,$$

$$c = \frac{a}{\cos \beta},$$

$$b = a \tan \beta,$$

$$A = \tfrac{1}{2} a^2 \tan \beta,$$

$$b = c \cos \alpha, \text{ check.}$$

$$\begin{aligned} \text{Sum of angles} &= 90^\circ\ 00' \\ \beta &= \underline{73^\circ\ 46'} \\ \alpha &= 16^\circ\ 14'. \end{aligned}$$

$$\begin{aligned} \log a &= 1.71366 \\ \log \cos \beta &= \underline{9.44646 - 10} \\ \log c &= 2.26720 \\ \therefore\ c &= 185.01 \end{aligned}$$

$$\begin{aligned}\log a &= 1.71366\\ \log\tan\beta &= \underline{0.53587}\\ \log b &= 2.24953\\ \therefore\ b &= 177.64\end{aligned}$$

. . . .

$$\begin{aligned}2\log a &= 3.42732\\ \log\tan\beta &= 0.53587\\ \operatorname{colog} 2 &= \underline{9.69897-10}\\ \log A &= 13.66216-10\\ \therefore\ A &= 4593.67\end{aligned}$$

Check

$$\begin{aligned}\log c &= 2.26720\\ \log\cos\alpha &= \underline{9.98233-10}\\ \log b &= 12.24953-10\\ \therefore\ b &= 177.64\end{aligned}$$

Note that $\log a^2 = 2\log a$. In solving, first write all the forms needed for the complete solution; secondly, look up and write in all the needed logarithms of the data from the tables; thirdly, perform the additions and subtractions; lastly, from the logarithmic results find the numbers. Then $\log\cos\beta$, $\log\tan\beta$, and $\log\cos\alpha\ (=\log\sin\beta)$ can all be found from once turning to the angle 73° 46′.

A form of computation sometimes used is given below. It has the advantage of being more compact than the usual form, and furthermore the logarithms of the data stand close to the data, thus permitting easy verification of results or correction of mistakes.

			Check
$a=51.72$	log 1.71366	log 1.71366	
$\beta=73°\,46'$	log cos $\underline{9.44646-10}$	log tan $\underline{0.53587}$	
$c=185.01$	log 2.26720		log 2.26720
$b=177.64$		log 2.24953	
$\alpha=16°\,14'$			log cos $\underline{9.98233-10}$
$b=177.64$			log 2.24953

a^2	log	3.42732
β	log tan	0.53587
2	colog	$\underline{0.69897-10}$
$A=4593.7$	log	3.66216

Example 2. Given $b = 7124.5$, $c = 9365.4$.

We have,

$$\begin{aligned} \cos\alpha &= \frac{b}{c}, \\ \beta &= 90^\circ - \alpha, \\ a &= c \sin\alpha, \\ A &= \tfrac{1}{2} bc \sin\alpha, \\ a &= b \tan\alpha, \text{ check.} \end{aligned}$$

$$\begin{aligned} \log b &= 3.85275 \\ \log c &= \underline{3.97153} \\ \log\cos\alpha &= 9.88122 - 10 \\ \alpha &= 40^\circ\ 28.4' \\ \beta &= 49^\circ\ 31.6' \end{aligned}$$

$$\begin{aligned} \log c &= 3.97153 \\ \log\sin\alpha &= \underline{9.81231 - 10} \\ \log a &= 13.78384 - 10 \\ a &= 6079.2 \end{aligned}$$

Check

$$\begin{aligned} \log b &= 3.85275 \\ \log\tan\alpha &= \underline{9.93109 - 10} \\ \log a &= 13.78384 - 10 \\ a &= 6079.2 \end{aligned}$$

The following is the compact arrangement of the computation:

			Check
$b = 7124.5$	log 3.85275		log 3.85275
$c = 9365.4$	log 3.97153	log 3.97153	
$\alpha = 40^\circ\ 28.4'$	log cos 9.88122 − 10	log sin 9.81231 − 10	log tan 9.93109 − 10
$\beta = 49^\circ\ 31.6'$			
$a = 6079.2$		log 3.78384	
$a = 6079.2$			log 3.78384

It appears that the Pythagorean proposition, $a^2 + b^2 = c^2$, is not used because it is not adapted to the use of logarithms. It might be used in this case, however, in the form

$$a = \sqrt{(c + b)(c - b)}.$$

EXERCISE XII

Find the missing parts of the following triangles, using logarithms. (The work may be checked with a slide rule.)

	α	β	a	b	c	A
1.	63°				2584	
2.			7531		8642	
3.	75° 15.2′		965.24			
4.			47.193			3972.6
5.			7.3298	6.1032		
6.	18° 25.5′				32.96	
7.			132.97			985.27
8.			53.215	13.712		
9.			65983		72916	
10.	29° 50.2′			10.207		
11.	25° 17.4′					382.97
12.			.00020	.00037		
13.		63° 12.7′			7.1436	
14.				.07154	.09127	
15.		35° 16.4′	.62961			
16.		35° 16.8′				41658
17.			.00615	.00415		
18.		80° 12.5′		5.2108		
19.				.00729	.01625	
20.		25° 18.2′			1729.3	

The examples 1–20 of Exercise VIII may also be solved by logarithms and the results compared with those there obtained.

21. Find the radius of the circle inscribed in a regular pentagon whose side is 12 feet.

22. Find the side of a regular pentagon inscribed in a circle whose radius is 15 feet 7 inches.

23. Find the area of a regular octagon whose circumscribed circle has a diameter of 10 feet.

24. A tower 120 feet high throws a shadow 69.2 feet long upon the plane of its base. What is the angle of inclination of the sun?

25. The top of a certain lighthouse is known to be 73 feet above the water. From a boat the angle between the top and its reflection is measured as 6° 45′. How far is the boat from the light?

26. Two trains leave a station at the same time, one going north at the rate of 30 miles per hour, and the other east at the rate of 40 miles per hour.

How far apart will they be in 20 minutes, and what is the direction of the line joining them?

27. Show that if a is the side of a regular polygon of n sides, the area of the polygon is given by $A = \frac{1}{4}a^2n \cot \frac{180°}{n}$.

28. Show that if r is the radius of a circle, then the area of a regular circumscribed polygon of n sides is $A = r^2n \tan \frac{180°}{n}$.

29. Find a value for the area of an inscribed polygon corresponding to that given above.

30. Taking the moon's diameter as 31′ 20″ and its distance from the earth as 239,000 miles, what is its diameter in miles?

31. At what distance may a mountain 4 miles high be seen across a plain, the earth being taken as a sphere of 4000 miles radius?

32. If the sun's diameter is taken at 866,000 miles and its distance from the earth as 93,000,000 miles, what angle should it subtend at the center of the earth.

33. An approximate formula for the distance from the midpoint of a circular arc to the midpoint of its chord is $m = \frac{l^2}{4}\frac{\sin \alpha}{100}$, in which l is the length of the chord in feet and α the deflection or circumferential angle subtended by a base or chord of 100 feet. Find m for $l = 30$, $\alpha = 2°$.

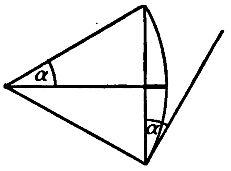

FIG. 21.

34. If I is the angle of intersection between two tangents to a circle of radius R, the distance T from a point of tangency to the point of intersection is given by $T = R \cot \frac{I}{2}$. Find T for $R = 3000$ feet, and $I = 22° 52′$.

35. The length of a chord is given by $2 R \sin \frac{1}{2} I$, in which I is the central angle. Find the chord length for $R = 2000$, $I = 12° 13′$.

36. A river which obstructs chaining on a survey is passed by triangulation. The line AB, Fig. 22, is measured 200 feet perpendicular to AC, and the angle ABC found to be 35° 27′. What is the distance AC?

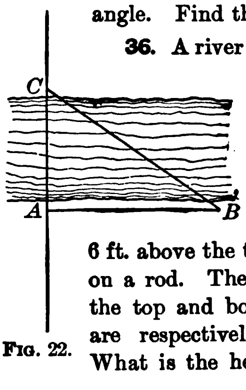

FIG. 22.

37. With an instrument at A, Fig. 23, a level line of sight passes 6 ft. above the top of a wall as measured on a rod. The angles of depression* to the top and bottom of the vertical face are respectively, 2° 31′ and 42° 16′. What is the height of the wall?

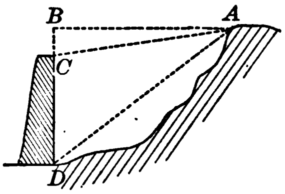

FIG. 23.

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* The angles of elevation and depression of an object measure respectively its angular distance above or below the horizon of the observer.

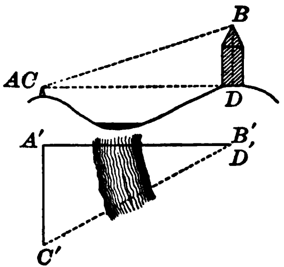

FIG. 24.

38. In order to obtain both the horizontal and vertical distances to an inaccessible point, the solution of two triangles may be necessary. Fig. 24 represents two views of the problem. Wishing the distances *AD* and *BD*, first lay out the base line *AC* of any convenient length perpendicular to *AB*. Measure the angle *ACD* and compute *AD*.

Next from *AD* and the angle *DAB*, the angles of elevation, compute *DB*.

Having $AC = 300$ ft., $ACB = 61° 34'$, and $DAB = 14° 41'$, find *AD* and *DB*.

CHAPTER V

THE OBTUSE ANGLE

32. Definitions of the trigonometric functions of obtuse angles. If an obtuse angle (*i.e.* an angle greater than 90° and less than 180°) is placed on the axes of coördinates in the same manner as was the acute angle in Art. 6, the terminal line will extend into the second quadrant. The trigonometric functions of such angles are defined exactly as in Art. 6. Thus in Fig. 25,

$$\sin \alpha = \frac{y}{v},$$

$$\cos \alpha = \frac{x}{v},$$

$$\tan \alpha = \frac{y}{x}, \text{ etc.}$$

33. Signs and limitations in value. The abscissas of all points in OA (Fig. 25) are negative, while their ordinates and radii vectores are positive. It is evident, therefore, that some of the defining ratios are negative. In accordance with the law of signs in algebraic division, we find that the sines and cosecants of all obtuse angles are positive, while their cosines, secants, tangents, and cotangents are negative.

The student should verify each of these statements in detail and become unhesitatingly familiar with these fundamental facts.

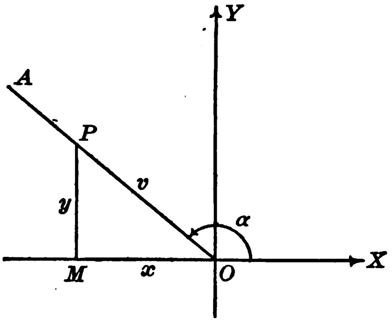

FIG. 25.

Furthermore, the sine and cosine cannot be numerically greater than unity and the secant and cosecant cannot be numerically less than unity.

QUERY. What are the limitations in value of the tangent and cotangent?

34. Fundamental relations. If the effects of the law of signs are traced, it will be seen that all the relations of Art. 9 hold also for functions of an obtuse angle without any modifications.

35. Variation. As the angle θ varies from 90° to 180°, while v remains constant, x is always negative and varies from 0 to $-v$, and y is positive and varies from v to 0. Consequently, as θ increases from 90° to 180°, $\sin\theta$ decreases from 1 to 0, $\cos\theta$ decreases (algebraically) from 0 to -1, $\tan\theta$ increases from $-\infty$ to 0, $\cot\theta$ decreases from 0 to $-\infty$, $\sec\theta$ increases from $-\infty$ to -1, $\csc\theta$ increases from 1 to ∞.

The terms positive infinity and negative infinity require careful consideration. If θ varies continuously from 89° to 90°, $\tan\theta$ varies in such a way as to exceed in magnitude any previously assigned definite value, however large. As it is positive for all values of θ in the first quadrant, it is consequently said to become positively infinite $(+\infty)$. If θ varies continuously from 91° to 90°, $\tan\theta$ varies so as to exceed numerically any previously assigned definite value. As it is, however, always negative for values of θ in the second quadrant, it is said to become negatively infinite $(-\infty)$. The plus or minus sign written before the symbol ∞ merely indicates whether the trigonometric function increases numerically without limit through a positive or a negative set of values.

36. Functions of 180°. As θ approaches 180°, v remaining constant, x approaches $-v$ and y approaches 0. We have, then,

$$\sin 180° = 0,$$

$$\cos 180° = -1,$$

$$\tan 180° = 0,$$

$$\cot 180° = \infty,$$

$$\sec 180° = -1,$$

$$\csc 180° = \infty.$$

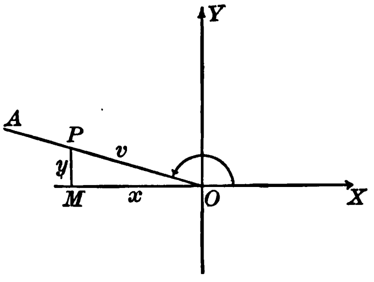

FIG. 26.

37. Functions of supplementary angles. Two angles are called supplementary if their sum is 180°. Thus, in Fig. 27, α and β are supplementary, and $\beta = 180 - \alpha$, α being acute. The triangles *OMP* and *ONQ* are similar, but *ON* is negative. The pairs of corresponding sides are v and v', x and x', y and y'. Hence we have

$$\sin(180° - \alpha) = \sin\beta = \frac{y'}{v'} = \frac{y}{v} = \sin\alpha,$$

$$\cos(180° - \alpha) = \cos\beta = \frac{x'}{v'} = -\frac{x}{v} = -\cos\alpha,$$

$$\tan(180° - \alpha) = \tan\beta = \frac{y'}{x'} = -\frac{y}{x} = -\tan\alpha.$$

Similarly:

$$\cot(180° - \alpha) = -\cot\alpha,$$

$$\sec(180° - \alpha) = -\sec\alpha,$$

$$\csc(180° - \alpha) = \csc\alpha.$$

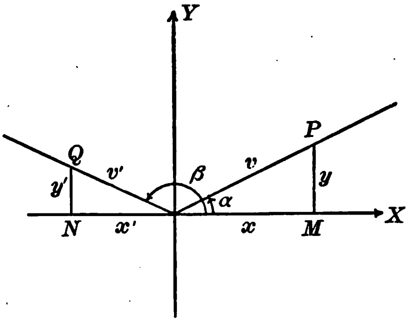

FIG. 27.

As a consequence of the relation $\sin(180° - \alpha) = \sin\alpha$, two values exist for arcsin m, the one acute, the other obtuse, and supplemental to each other. In case $m = 1$, the two values are identical.

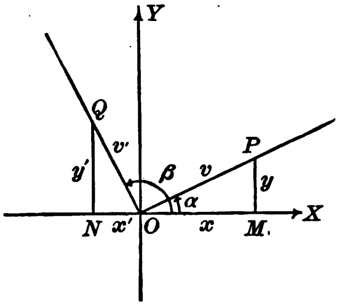

FIG. 28.

QUERY. Is this also true of arccos m, arctan m, etc.?

38. Functions of $(90° + \alpha)$. In Fig. 28, $\beta = 90° + \alpha$, α being acute. The triangles *OMP* and *ONQ* are similar, but the pairs of homologous sides are v and v', x and y', y and x', while x' is negative. We thus obtain

$$\sin(90° + \alpha) = \sin\beta = \frac{y'}{v'} = \frac{x}{v} = \cos\alpha,$$

$$\cos(90° + \alpha) = \cos\beta = \frac{x'}{v'} = -\frac{y}{v} = -\sin\alpha,$$

$$\tan(90° + \alpha) = \tan\beta = \frac{y'}{x'} = -\frac{x}{y} = -\cot\alpha.$$

In like manner,

$$\cot(90^\circ + \alpha) = -\tan\alpha,$$
$$\sec(90^\circ + \alpha) = -\csc\alpha,$$
$$\csc(90^\circ + \alpha) = \sec\alpha.$$

EXERCISE XIII

1. Find the values of the functions of 135°. (See Art. 11.)

2. Find the values of the functions of 150°. (See Art. 11.)

3. Find the value of $\sin[\cos^{-1}(-\frac{33}{65})]$, $\tan(\csc^{-1}\frac{61}{11})$, $\cos[\arctan(-\frac{9}{40})]$, the angles being of the second quadrant.

4. Find the value of $\cos(\arccos -\frac{9}{41})$, $\sin[\tan^{-1}(-\frac{15}{112})]$, $\cot(\arcsin\frac{63}{65})$, the angles being of the second quadrant.

5. Express in terms of an angle less than 45°, cos 160°, tan 130°, sec 150°.

6. Express in terms of an angle less than 45°, sin 170°, csc 95°, cot 140°.

7. Verify for $\alpha = 60^\circ$, the equations

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \quad \cos 2\alpha = 2\cos^2\alpha - 1.$$

8. Verify for $\alpha = 45^\circ$, the equations

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha,$$
$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha.$$

9. Verify for $\alpha = 120^\circ$, the equations

$$\cos\frac{1}{2}\alpha = \sqrt{\frac{1+\cos\alpha}{2}},$$
$$\tan\frac{1}{2}\alpha = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha}.$$

10. Verify for $\alpha = 120^\circ$, the equations

$$\sin\frac{1}{2}\alpha = \sqrt{\frac{1-\cos\alpha}{2}},$$
$$\cot\frac{1}{2}\alpha = \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} = \frac{1+\cos\alpha}{\sin\alpha}.$$

11. Verify for $\alpha = 120^\circ$, $\beta = 30^\circ$, the equations

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

12. Verify for $\alpha = 120^\circ$, $\beta = 60^\circ$, the equations

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

13. Fill in the proper values in the following table for handy reference:

a	$\sin a$	$\cos a$	$\tan a$	$\cot a$	$\sec a$	$\csc a$	
0°	0						
30°	$\frac{1}{2}$						
45°	$\frac{1}{2}\sqrt{2}$						
60°	$\frac{1}{2}\sqrt{3}$						
90°	1						
120°	$\frac{1}{2}\sqrt{3}$						
135°	$\frac{1}{2}\sqrt{2}$						
150°	$\frac{1}{2}$						
180°	0						

CHAPTER VI

OBLIQUE TRIANGLES

39. Formulas for solution. In the oblique triangle *ABC*, Fig. 29, let the angles be denoted by α, β, γ, and the lengths of the opposite sides by a, b, c, as in the figure.

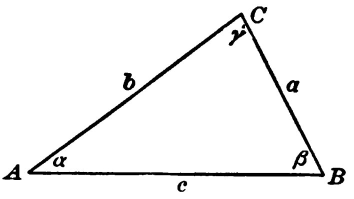

FIG. 29.

The relation $\boldsymbol{\alpha + \beta + \gamma = 180°}$ always exists, and consequently when two of the angles are known, the third is determined. Five of the six parts of the triangle still remain to be found; namely, the three sides and two angles. It has been shown in elementary geometry that if any three independent parts are given, the triangle is determined and the remaining parts can be found. Then two formulas, in addition to the one just stated, are sufficient for the complete solution.

It is, nevertheless, convenient to express the relations between the sides and angles in a variety of forms. Those given in the following pages are selected on the score of utility. They fall into sets of three each. From any one of each set the other two may be written by cyclic advance of the letters involved; *i.e.* by changing a into b, b into c, c into a, and at the same time α into β, β into γ, γ into α. The legitimacy of this process and the truth of the resulting formulas appear from the consideration that no distinction is made as to any one side or any one angle. Any side and its opposite angle can be exchanged for any other pair. The cyclic advance affords a convenient systematic method of writing all possible forms.

From any one of these sets, as for instance that of Art. 40, or that of Art. 42, all the other sets may be derived by purely analytical processes. An independent geometric proof is given of each, however. The derivation by the analytic method suggested will afford a valuable review exercise after Chapter VIII has been studied.

40. Law of projections. If, in Fig. 30, the perpendicular CD is drawn from C to AB, the portions AD and DB are respectively the projections on the side AB of the other two sides AC and CB. Consequently, by Art. 15, we have

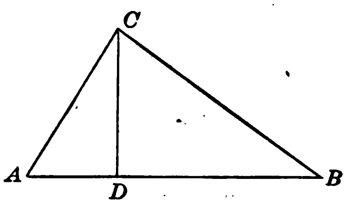

FIG. 30.

$$AB = AC \cos \alpha + CB \cos \beta,$$

or
$$c = b \cos \alpha + a \cos \beta.$$

By drawing the perpendicular from A and B in turn, we get

$$a = c \cos \beta + b \cos \gamma,$$
$$b = a \cos \gamma + c \cos \alpha.$$

By cyclic advance of the letters the first formula is transformed into the second, the second into the third, and the third into the first.

41. Law of sines. Connect the circumcenter K in Fig. 31 with the vertices, A, B, C, and the midpoints, L, M, N, of the sides. Then is $\angle BKC = 2\alpha$, $\angle CKA = 2\beta$, $\angle AKB = 2\gamma$. (Why?) In the right triangle KLC, $\angle LKC = \alpha$, and $LC = \frac{1}{2}a$. Denoting the circumradius by R, Art. 16 gives

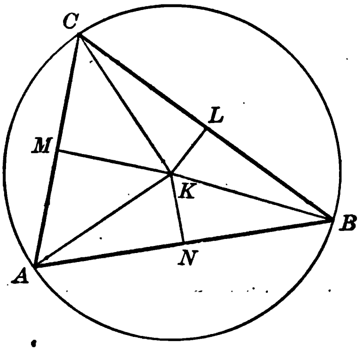

FIG. 31.

$$\tfrac{1}{2}a = R \sin \alpha.$$

The other right triangles give likewise

$$\tfrac{1}{2}b = R \sin \beta,$$
$$\tfrac{1}{2}c = R \sin \gamma.$$

Equating the values of $2R$, we obtain the law of sines; namely,

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

The cyclic symmetry is apparent.

The student should draw the figure and give the proof in case one angle of the triangle is obtuse.

42. Law of cosines. In Fig. 32 the perpendicular p drawn from C divides the opposite side c into two portions m and n, and the whole triangle into two right triangles ADC and BDC. In the latter triangles, we have, by Art. 16,

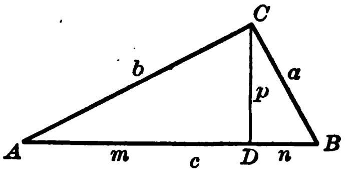

FIG. 32.

$$a^2 = n^2 + p^2$$
$$= (c - m)^2 + p^2$$
$$= (c - b\cos\alpha)^2 + b^2\sin^2\alpha;$$

or $\boldsymbol{a^2 = b^2 + c^2 - 2\,bc\cos\alpha}$.

Proper changes in the figure yield

$$\boldsymbol{b^2 = c^2 + a^2 - 2\,ac\cos\beta,}$$
$$\boldsymbol{c^2 = a^2 + b^2 - 2\,ab\cos\gamma.}$$

These again may be written by cyclic advance of the letters. Useful forms for writing these laws are:

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2\,bc},$$

$$\cos\beta = \frac{c^2 + a^2 - b^2}{2\,ac},$$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2\,ab}.$$

43. Law of tangents. In Fig. 33 draw AE the bisector of the angle at A, and BF and CD perpendicular to it from the other vertices.

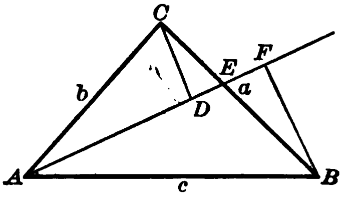

FIG. 33.

Then

$$\angle BAF = \angle DAC = \tfrac{1}{2}\alpha,$$

while

$$\angle DCE = \angle EBF = 90^\circ - \angle BEF$$
$$= 90^\circ - (\angle ABE + \angle BAE)$$
$$= \tfrac{1}{2}(\alpha + \beta + \gamma) - (\beta + \tfrac{1}{2}\alpha)$$
$$= \tfrac{1}{2}(\gamma - \beta).$$

Again,

$$DF = EF + DE = AF - AD.$$

From the right triangles in the figure we get

$$\tan\tfrac{1}{2}(\gamma - \beta) = \frac{EF}{FB} = \frac{DE}{CD} = \frac{EF + DE}{FB + CD} = \frac{AF - AD}{FB + CD} = \frac{(c-b)\cos\frac{1}{2}\alpha}{(c+b)\sin\frac{1}{2}\alpha}$$

or $$\boldsymbol{\tan\tfrac{1}{2}(\gamma - \beta) = \frac{c - b}{c + b}\cot\tfrac{1}{2}\alpha.}$$

The forms
$$\tan \tfrac{1}{2}(\alpha - \gamma) = \frac{a-c}{a+c} \cot \tfrac{1}{2}\beta,$$
$$\tan \tfrac{1}{2}(\beta - \alpha) = \frac{b-a}{b+a} \cot \tfrac{1}{2}\gamma,$$
may be obtained from suitably altered figures or by cyclic advance.

The formula may be written symmetrically
$$\frac{\tan \frac{1}{2}(\gamma - \beta)}{\tan \frac{1}{2}(\gamma + \beta)} = \frac{c-b}{c+b}.$$

If $b > c$, the first formula will stand
$$\tan \tfrac{1}{2}(\beta - \gamma) = \frac{b-c}{b+c} \cot \tfrac{1}{2}\alpha.$$

Similar changes may occur in the other two.

44. Angles in terms of the sides. Construct the inscribed circle, Fig. 34, and denote its radius by r. Denoting the perimeter $a + b + c$ by $2s$, we have

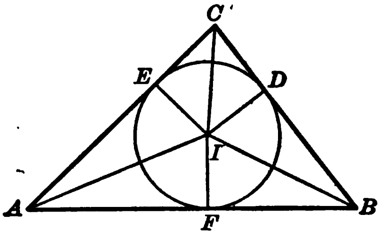

FIG. 34.

$$AE = AF = s - a,$$
$$BD = BF = s - b,$$
$$CD = CE = s - c.$$

Consequently, by Art. 16,
$$\tan \tfrac{1}{2}\alpha = \frac{r}{s-a}, \quad \tan \tfrac{1}{2}\beta = \frac{r}{s-b}, \quad \tan \tfrac{1}{2}\gamma = \frac{r}{s-c}.$$

The value of r in terms of the three sides is derived in the corollary of Art. 45, thus completing this theorem.

45. Area of oblique triangles.

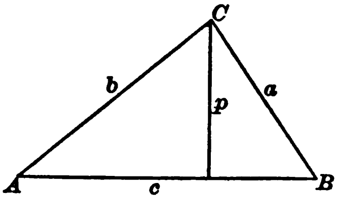

FIG. 35.

(1) By elementary geometry, we have (see Fig. 35)
$$A = \tfrac{1}{2}pc.$$

Introducing the value of p found by Art. 16, we get the formula
$$A = \tfrac{1}{2}bc \sin \alpha,$$

with the cognate forms

$$A = \tfrac{1}{2} ca \sin \beta, \quad A = \tfrac{1}{2} ab \sin \gamma.$$

(2) Squaring both members of the formula just derived, we obtain, with the aid of readily justifiable transformations and substitutions,

$$\begin{aligned}
A^2 &= \tfrac{1}{4} b^2c^2 \sin^2 \alpha \\
&= \tfrac{1}{4} b^2c^2(1 - \cos^2 \alpha) \\
&= \frac{bc}{2}(1 + \cos \alpha) \cdot \frac{bc}{2}(1 - \cos \alpha) \\
&= \frac{bc}{2}\left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right) \cdot \frac{bc}{2}\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right) \\
&= \frac{2bc + b^2 + c^2 - a^2}{4} \cdot \frac{2bc - b^2 - c^2 + a^2}{4} \\
&= \frac{b + c + a}{2} \cdot \frac{b + c - a}{2} \cdot \frac{a - b + c}{2} \cdot \frac{a + b - c}{2} \\
&= s(s - a)(s - b)(s - c).
\end{aligned}$$

Whence we have the desired formula

$$A = \sqrt{s(s - a)(s - b)(s - c)}.$$

(3) If r is the radius of the inscribed circle, we have, by elementary geometry,

$$A = rs.$$

Corollary. Equating the values of A found in (2) and (3), and solving for r, we get

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

the result needed to complete the theorem of Art. 44.

46. Numerical solution. The formulas of Arts. 40 and 42 are not adapted to the employment of logarithms. They are useful, however, in case the numerical values of the sides contain few digits.

The solution of oblique triangles falls into four well-defined cases, according as the three given parts consist of

I. Two angles and one side.
II. Two sides and an angle opposite one of them.
III. Two sides and the included angle.
IV. Three sides.

Each of these three cases with a model solution is discussed in detail in the following articles.

47. CASE I. **Given two angles and one side.** Let the given parts be α, β, a.

The solution is effected by means of the formulas of Arts. 39 and 41. Solving for the unknown parts, we have

$$\gamma = 180^\circ - (\alpha + \beta),$$

$$b = \frac{a \sin \beta}{\sin \alpha},$$

$$c = \frac{a \sin \gamma}{\sin \alpha},$$

with the check formula

$$c = \frac{b \sin \gamma}{\sin \beta}.$$

EXAMPLE. Given

$$\alpha = 47^\circ\ 13.2'$$
$$\beta = 65^\circ\ 24.5'$$
$$a = 43.176$$

$$\begin{aligned} \text{sum of angles} &= 180^\circ \\ \alpha + \beta &= \underline{112^\circ\ 37.7'} \\ \therefore\ \gamma &= 67^\circ\ 22.3' \end{aligned}$$

$$\begin{aligned} \log a &= 1.63524 \\ \log \sin \beta &= 9.95871 - 10 \\ \text{colog} \sin \alpha &= \underline{0.13433} \\ \log b &= 11.72828 - 10 \\ \therefore\ b &= 53.491 \end{aligned}$$

$$\begin{aligned} \log a &= 1.63524 \\ \log \sin \gamma &= 9.96522 - 10 \\ \text{colog} \sin \alpha &= \underline{0.13433} \\ \log c &= 11.73479 - 10 \\ \therefore\ c &= 54.299 \end{aligned}$$

Check

$$\begin{aligned} \log b &= 1.72828 \\ \log \sin \gamma &= 9.96522 - 10 \\ \text{colog} \sin \beta &= \underline{0.04129} \\ \log c &= 11.73479 - 10 \\ \therefore\ c &= 54.299 \end{aligned}$$

The compact form of computation is as follows:

			Check
$a = 43.176$	log 1.63524	log 1.63524	
$\beta = 65° \, 24.5'$	log sin 9.95871 − 10		colog sin 0.04129
$\alpha = 47° \, 13.2'$	colog sin 0.13433	colog sin 0.13433	
$b = 53.491$	log b 1.72828		log 1.72828
$\gamma = 67° \, 22.3'$		log sin 9.96522 − 10	log sin 9.96522 − 10
$c = 54.299$		log c 1.73479	log c 1.73479
$c = 54.299$			

EXAMPLES

Find the remaining three parts, given

1. $\beta = 65° \, 15.5'$, $\gamma = 81° \, 24.6'$, $b = 724.32$.
2. $\beta = 38° \, 37.4'$, $\gamma = 75° \, 32.8'$, $c = 129.63$.
3. $\alpha = 48° \, 29.2'$, $\gamma = 115° \, 33.8'$, $a = 14.829$.
4. $\alpha = 68° \, 41.5'$, $\gamma = 110° \, 16.5'$, $c = 9.4326$.

48. CASE II. **Given two sides and an angle opposite one.** Let the given parts be a, b, α.

The solution is effected by the formulas of Arts. 39 and 41. Solving, we have

$$\sin \beta = \frac{b \sin \alpha}{a},$$

$$\gamma = 180° - (\alpha + \beta),$$

$$c = \frac{a \sin \gamma}{\sin \alpha},$$

with the check formula

$$c = \frac{b \sin \gamma}{\sin \beta}.$$

An ambiguity arises in this case, however, since to any value of the sine correspond two supplementary angles, one acute, the other obtuse. Thus we also have

$$\beta' = 180° - \beta,$$

$$\gamma' = 180° - (\alpha + \beta'),$$

$$c' = \frac{a \sin \gamma'}{\sin \alpha},$$

$$c' = \frac{b \sin \gamma'}{\sin \beta'}.$$

The nature of this ambiguity will appear from the construction of the triangles with the given parts. If the given angle α is acute, there will be no solution, one solution, or two solutions, according as the free end of a (see Fig. 36), swinging about C, meets the line AL in no points, one point, or two points; *i.e.* as a is shorter than CD, the perpendicular from C upon AL, longer than AC, or intermediate between CD and AC. For $a = CD$ there is a single right triangle; and for $a = AC$, a single isosceles triangle.

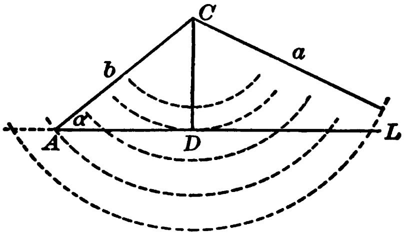

Fig. 36.

When α is right or obtuse, there is no solution or one solution, according as a is shorter or longer than AC.

These results may be tabulated for reference.

$$\alpha < 90^\circ \begin{cases} a < b \sin \alpha, & \text{no solution,} \\ b \sin \alpha < a < b, & \text{two solutions,} \\ \left.\begin{matrix} a \geqq b, \\ a = b \sin \alpha, \end{matrix}\right\} & \text{one solution.} \end{cases}$$

$$\alpha \geqq 90^\circ \begin{cases} a \leqq b, & \text{no solution,} \\ a > b, & \text{one solution.} \end{cases}$$

If we proceed with the numerical work, without previously testing the number of solutions possible, the case of a single solution will appear from the fact that $\alpha + \beta' > 180^\circ$. (Whence $\alpha + (180^\circ - \beta) > 180^\circ$, or $\alpha - \beta > 0$, or $\beta < \alpha$.) When there is no solution, we shall get $\log \sin \beta > 0$; *i.e.* its augmented characteristic will be 10 or greater. A preliminary free-hand sketch will ordinarily serve to determine the number of possible solutions.

Example 1. Given

$$a = 3541,$$
$$b = 4017,$$
$$\alpha = 61^\circ\ 27'.$$

By careful arrangement of the work, we can determine the number of solutions by inspection.

			Check
$b=4017$	log 3.60390		log 3.60390
$\alpha=61° 27'$	log sin 9.94369 − 10	colog sin 0.05631	
$b \sin \alpha$	log 3.54759		
$a=3541$	log 3.54913	log 3.54913	
$\beta=85° 11'$	log sin 9.99846		colog sin 0.00154
$\alpha+\beta=146° 38'$			
$\gamma=33° 22'$		log sin 9.74036 − 10	log sin 9.74036 − 10
$c=2217.16$		log 3.34580	
$c=2217.16$			log 3.34580

From the logarithms of b, a, and $b \sin \alpha$ it is seen that $b \sin \alpha < a < b$, whence there are two solutions. For the second solution we have:

		Check
$\alpha=61° 27'$	colog sin 0.05631	
$\beta'=94° 49'$		colog sin 0.00154
$\alpha+\beta'=156° 16'$		
$\gamma'=23° 44'$	log sin 9.60474 − 10	log sin 9.60474 − 10
$a=3541$	log 3.54913	
$b=4017$		log 3.60390
$c'=1622.52$	log 3.21018	
$c'=1622.52$		log 3.21018

EXAMPLE 2. How many triangles are determined by the given parts $\alpha = 30°$, $b = 24$, $a = 10, 12, 20, 24, 30$?

Here $b \sin \alpha = 24 \times \frac{1}{2} = 12$. Accordingly, we have, for $a = 10$, no triangle; for $a = 12$, one right triangle; for $a = 20$, two triangles; for $a = 24$, one isosceles triangle; and for $a = 30$, one triangle.

EXAMPLES

1. How many triangles are determined by the given parts $\beta = 43°$, $c = 120$, and $b = 63, 81.884, 95, 120, 150$?

2. How many triangles are determined by the given parts $\gamma = 54°$, $a = 75$, and $c = 51, 60, 67.5, 70, 75, 100$?

Find the remaining parts of all possible triangles, given

3. $a = 62.518$, $b = 72.932$, $\beta = 98° 23.5'$.
4. $a = 429.15$, $c = 328.12$, $\alpha = 130° 33.7'$.
5. $b = 3912.7$, $c = 3526.5$, $\gamma = 35° 25.8'$.
6. $b = 129680$, $c = 152960$, $\beta = 38° 28.8'$.

49. Case III. **Given two sides and the included angle.** Let the given parts be a, b, γ, with $a > b$. The solution is effected by the formulas of Arts. 43, 39, and 41. Solving, we have

$$\tan \tfrac{1}{2}(\alpha - \beta) = \frac{a-b}{a+b} \cot \tfrac{1}{2}\gamma,$$

$$\tfrac{1}{2}(\alpha + \beta) = 90^\circ - \tfrac{1}{2}\gamma,$$

$$c = \frac{a \sin \gamma}{\sin \alpha};$$

with the check formula

$$c = \frac{b \sin \gamma}{\sin \beta}.$$

Example. Given

$$a = .745,$$
$$b = .231,$$
$$\gamma = 78^\circ\ 15'.$$

$\gamma = 78^\circ\ 15'$		log sin 9.99080 − 10	log sin 9.99080 − 10
$a = .745$		log 9.87216 − 10	
$b = .231$			log 9.36361 − 10
$a - b = .514$	log 9.71096 − 10		
$a + b = .976$	colog 0.01055		
$\frac{\gamma}{2} = 39^\circ\ 7.5'$	log cot 0.08969		
$\frac{\alpha-\beta}{2} = 32^\circ\ 55.3'$	log tan 9.81120 − 10		
$\frac{\alpha+\beta}{2} = 50^\circ\ 52.5'$			
$\alpha = 83^\circ\ 47.8'$		colog sin 0.00255	
$\beta = 17^\circ\ 57.2'$			colog sin 0.51109
$c = .73368$		log 19.86551 − 20	
$c = .73367$			19.86550 − 20

Examples

Find the unknown parts, given

1. $b = 284.12$, $c = 361.26$, $\alpha = 125^\circ\ 32'$.
2. $c = 395.71$, $a = 482.33$, $\beta = 137^\circ\ 21'$.
3. $a = .06351$, $c = .10329$, $\beta = 83^\circ\ 29.4'$.
4. $c = .00397$, $b = .00513$, $\alpha = 68^\circ\ 21.8'$.

50. Case IV. **Given the three sides.** The given parts are a, b, c.

The solution is effected by the formulas of Art. 44, with the formula for r from Art. 45. We have at once

$$s=\tfrac{1}{2}(a+b+c),$$

$$r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

$$\tan\frac{1}{2}\alpha=\frac{r}{s-a},\ \text{etc.}$$

$$\alpha+\beta+\gamma=180^\circ,\ \text{serves as a check formula.}$$

Example 1. Given

$a=.05341,$
$b=.06217,$
$c=.03482.$

Then	$2s=.15040$		
	$s=.07520$	colog	1.12378
	$s-a=.02179$	log	$8.33826-10$
	$s-b=.01303$	log	$8.11494-10$
	$s-c=.04038$	log	$8.60617-10$
	r^2	log	$16.18315-20$
	r	log	$8.09157-10$
	$\frac{\alpha}{2}=29^\circ\ 32.3'$	log tan	$9.75331-10$
	$\frac{\beta}{2}=43^\circ\ 27.6'$	log tan	$9.97663-10$
	$\frac{\gamma}{2}=17^\circ\ 0.1'$	log tan	$9.48540-10$
	$\alpha=59^\circ\ 4.6'$		
	$\beta=86^\circ\ 55.2'$		
	$\gamma=34^\circ\ 0.2'$		
	sum of angles $=180^\circ\ 0'$		

When the three sides are given and only one angle is required, say β, the two appropriate formulas may be combined into one, as

$$\tan\frac{1}{2}\beta=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}.$$

Example 2. Given

$$\begin{array}{rl} a = & 35, \\ b = & 64, \\ c = & 73. \end{array}$$

Then

$$\begin{array}{rlrl} 2s = & 172 & & \\ s = & 86 & \text{colog} & 8.06550-10 \\ s-a = & 51 & \log & 1.70757 \\ s-b = & 22 & \text{colog} & 8.65758-10 \\ s-c = & 13 & \log & \underline{1.11394} \\ & & & 2\overline{)19.54459-20} \\ \frac{1}{2}\beta = & 30^\circ\, 37.4' & \log\tan & 9.77230-10 \\ \beta = & 61^\circ\, 14.8' & & \end{array}$$

Examples

Find the angles of the following triangles:

1. $a = 6123,\quad b = 7148,\quad c = 6815.$
2. $a = 12{,}545,\quad b = 8612,\quad c = 10{,}353.$
3. $a = .05431,\quad b = .03714,\quad c = .06513.$
4. $a = .006152,\quad b = .008174,\quad c = .007534.$
5. $a = 72{,}584,\quad b = 125{,}217,\quad c = 36{,}925.$
6. $a = 13{,}579,\quad b = 35{,}791,\quad c = 24{,}680$; find β.
7. $a = 80{,}812,\quad b = 37{,}194,\quad c = 43{,}618.$
8. $a = 36{,}925,\quad b = 25{,}814,\quad c = 14{,}703$; find γ.

Find the areas in examples 1 and 2.

51. Composition and resolution of forces. Equilibrium. In mechanics the solution of oblique triangles is frequently required in problems relating to the composition and resolution of forces, velocities, and other directed quantities.

In this article will be stated, without proof, some of the laws governing the combination of such quantities, showing the application of trigonometry to certain of the problems involved.

Suppose the line segments AB and AC, Fig. 37, to represent in magnitude and direction two forces acting at a point A, and including between their lines of action the angle ϕ.

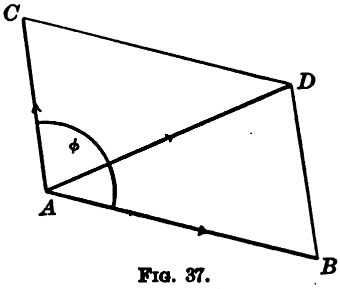

FIG. 37.

Complete the parallelogram $ABDC$. The diagonal AD, drawn from the point A, is the line segment representing the ***resultant*** of the two given forces, *i.e.* the single force that will produce the same effect as the two given forces. The process of finding the resultant of two or more given forces is called the *composition* of forces.

Conversely, the two line segments AB and AC may be taken as the components of AD. Thus the two forces AB and AC, acting together at A, produce the same effect as the single force AD. The process of finding two or more forces equivalent to a given force is called the ***resolution*** of the force into its components.

Since the segment BD is equal and parallel to AC, it follows that the resultant and the two components form a closed triangle ABD, and the relation between the forces may be obtained by solving this triangle. Note that the angle ABD is the supplement of the angle ϕ, so that by Art 37,

$$\cos ABD = -\cos \phi.$$

EXAMPLE 1. Find the resultant of two forces of 320 dynes and 400 dynes, respectively, acting on a common point, at an angle of 54° 28′.

In the triangle ABD, Fig. 37, we have given two sides and the included angle. If only the magnitude of the resultant is desired, it may be obtained by the law of cosines, Art. 42. Thus we obtain

$$AD = \sqrt{\{\overline{AB}^2 + \overline{AC}^2 + 2\,AB \cdot AC \cdot \cos \phi\}}.$$

If the angle formed by the resultant with its components is also required, the logarithmic computation may be effected as in Case III, Art. 49.

EXAMPLE 2. Resolve a force of 40 pounds into components making angles of 32° and 74° 20 with its line of action.

Referring to Fig. 37, we have

$$AD = 40, \angle BAD = 32°, \text{ and } \angle DAC = \angle ADB = 74° \, 20'.$$

Denoting the sides opposite the angles A, B, D, respectively, by a, b, d, we have from the law of sines,

$$a = b\frac{\sin A}{\sin B}, \qquad d = b\frac{\sin D}{\sin B}.$$

Hence the components may be computed.

Three forces are in equilibrium when the resultant of any two forces is equal and opposite to the third. Thus in Fig. 37, if the direction of the force AD is reversed, it and the forces AB and AC will be in equilibrium. The necessary conditions that three forces shall be in equilibrium are:

1. Their lines of action shall lie in the same plane.
2. Their lines of action shall meet in a point.
3. The line segments representing the three forces when laid off in order shall form a triangle.

In Fig. 38 the forces a, b, and c applied at a common point are in equilibrium. The angles between the lines of action are denoted by A, B, C, as indicated. When the forces are laid off to form the triangle, the angles of the triangle are seen to be the supplements of the corresponding angles A, B, C.

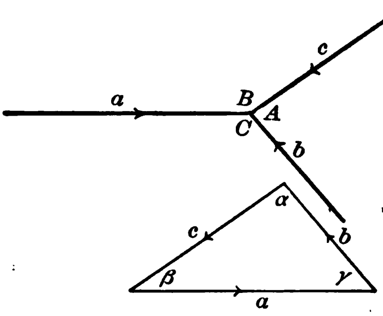

FIG. 38.

That is,

$$\alpha = 180^\circ - A, \text{ whence } \sin\alpha = \sin A,$$

$$\beta = 180^\circ - B, \text{ whence } \sin\beta = \sin B.$$

etc. etc.

From the law of sines,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Therefore,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

EXERCISE XIV

Find the unknown parts of the following triangles:

	α	β	γ	a	b	c
1.	62° 35′				82916	59278
2.				75290	92841	69289
3.	25° 36.2′	68° 13.5′		3.9168		
4.		55° 55.4′		.25317		.36291
5.	69° 17.5′			329.12	689.12	
6.		100° 10′			62198	29322
7.				.0000713	.0000987	.0001255
8.	61° 15.2′	49° 16.3′			58.291	
9.			120° 50.2′	2.8315	4.1217	
10.		38° 17.2′		21.992	50.715	
11.	150° 24.2′				.038251	.047319
12.	58° 06.5′			57.15		67.31
13.		75° 19.3′	70° 29.2′		658.42	
14.				100.05	200.07	150.08
15.		126° 26.4′		.0021868		.0032292
16.			10° 32.8′		25.317	37.293
17.				50010	70020	90030
18.		48° 25.3′	56° 34.5′			7219.2
19.			120° 15′	62158		75292
20.			90° 00′	725.63	617.25	

Solve the following triangles, given

21. $a = 2500$, $c = 2125$, $A = 208{,}690$.

22. $b = 103.5$, $c = 90$, $A = 4586.7$.

23. $\alpha = 73° 10'$, $b = 753$, $A = 74{,}803$.

24. $\beta = 57° 25'$, $c = 57.65$, $A = 3055.7$.

25. Find the areas in examples 1, 9, 17.

26. Find the areas in examples 2, 4, 14.

27. Determine the magnitude and direction of the resultant of two forces of magnitudes a and b, if their lines of action include an angle ϕ.

28. Carry out the computation of example 27 in the following cases:

$a = 20$, $b = 36$, $\phi = 45°$; $a = 300$, $b = 540$, $\phi = 64°$;
$a = 75$, $b = 60$, $\phi = 145°$; $a = 250$, $b = 320$, $\phi = 120°$.

29. Find the directions of three forces in equilibrium if $a = 7$, $b = 10$, $c = 15$; also if $a = 24$, $b = 36$, $c = 42$.

30. Referring to Figure 38 solve completely and interpret physically when $a = 695$, $b = 483$, $C = 155°$; $a = 720$, $b = 840$, $B = 100°$.

31. Solve and interpret when $a = 1200$, $B = 135°$, $C = 150°$; $a = 135$, $b = 142$, $c = 95$.

32. Resolve a force of magnitude 84 into two equal components making an angle of 60° with each other.

33. Resolve a force of magnitude 240 into two components of 120 and 180 each and find the directions of the components.

34. Determine the formula for one side of a quadrilateral in terms of the other three sides and their included angles. Compute for $a = 10$, $b = 12$, $c = 15$, $\widehat{ab} = 135°$, $\widehat{bc} = 60°$.

QUERY. How many given parts serve to determine the remaining parts of a quadrilateral?

35. Given the four sides and one angle of a quadrilateral, determine the other angles and the diagonals. Compute for $a = 60$, $b = 72$, $c = 90$, $d = 100$, $\widehat{ab} = 120°$.

36. Given three angles and two sides of a quadrilateral, determine the remaining sides. Compute for $a = 630$, $b = 500$, $\widehat{ab} = 100°$, $\widehat{bc} = 80°$, $\widehat{cd} = 60°$.

37. Find the angles and the lengths of the sides of a regular pentagram, or five-pointed star, inscribed in a circle of radius 8.

38. Compute the volume for each foot in depth of a horizontal cylindrical tank of length 30 feet and radius 6 feet.

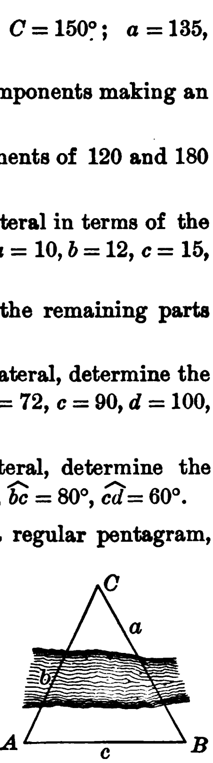

FIG. 39.

39. Having measured the following data, $A = 80° 30'$, $B = 72° 15'$, and $c = 232.5$ feet, compute the inaccessible distance b (Fig. 39).

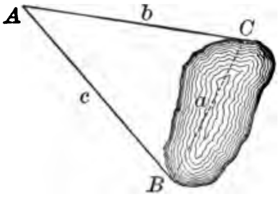

FIG. 40.

40. Compute the distance a across a lake, Fig. 40, having measured A, B, and c, which are respectively 51° 20′, 72° 40′ and 3420.5 feet.

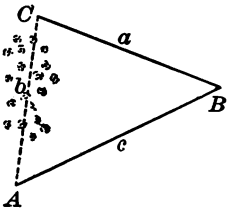

FIG. 41.

41. A being invisible from C, find the distance b through a forest, having measured $a = 1037$ feet, $c = 1208$ feet, $B = 69° 25'$.

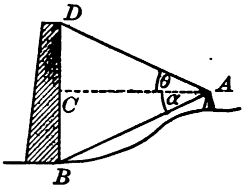

FIG. 42.

42. In Fig. 42, BC, the distance of the foot of a wall below the instrument is 12.3 feet, θ and α, the angles of elevation and depression, are 15° 20′ and 21° 15′, respectively. Find the height of the wall and its distance from the instrument.

43. A pole BC, Fig. 43, is 12 feet long and leans two feet from a vertical toward the instrument at A. If the angles of elevation of the top and bottom are respectively 37° 15′ and 11° 50′, what are the horizontal and vertical distances from the instrument to the foot of the pole?

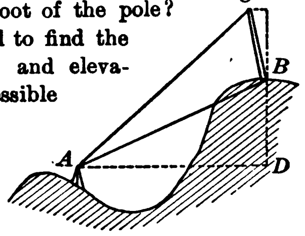

FIG. 43.

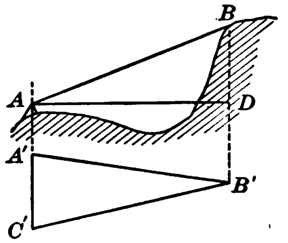

FIG. 44.

44. It is desired to find the horizontal distance and elevation of the inaccessible point B, Fig. 44, with reference to an instrument at A. Having laid out a base line AC, 250 feet long, the angles at A and C are found to be 87° 10′ and 73° 51′, respectively, and from A the angular elevation of B is 11° 32′.

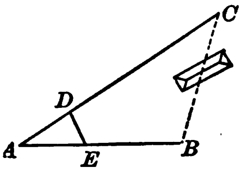

FIG. 45.

45. Given $B = 110° \ 05'$, $AE = AD = 200$ feet, $DE = 125$ feet, and $AB = 632$ feet; find the distance AC to be laid off, and the inaccessible distance BC (Fig. 45).

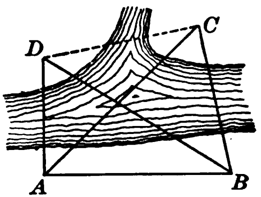

FIG. 46.

46. From measurements we have (Fig. 46) $AB = 600$ feet, $BAC = 70° \ 40'$, $BAD = 92° \ 30'$, $ABD = 65° \ 32'$, $ABC = 89° \ 25'$. Find the inaccessible distances AD and DC, and the angle between DC and AB.

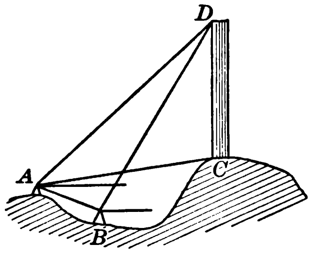

FIG. 47.

47. From the instrument at A (Fig. 47) the angles of elevation to the top and base of the vertical wall are 15° 12′ and 1° 23′, respectively. A base line AB is measured 75 feet toward the wall down a plane inclined 8° 16′, and from B the angle of elevation to the top of the wall is 37° 46′. Compute the height of the wall and its horizontal distance from A.

48. It is required to prolong the line AB (Fig. 48) beyond an obstacle. At B is made an angle 52° 20′ to the right and at C an angle of 110° 00′ to the left, BC being 210 feet. Compute the proper distance CD and angle to the right at D, also the inaccessible distance BD. Note that by making $B = D = 60°$ and $C = 120°$, then $BC = CD = BD$ and all computations are avoided.

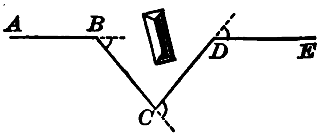

FIG. 48.

49. Having but one point C (Fig. 49) from which both inaccessible points A and B are visible, we are required to find the inaccessible distances AC

and AD and the angle between AB and DC. $ADC = 87^\circ\ 42'$, $DCA = 60^\circ\ 32'$, $DCE = 170^\circ\ 05'$, $BCE = 41^\circ\ 20'$, $CEB = 111^\circ\ 35'$, $DC = 365.2$ feet, $CE = 410.7$ feet.

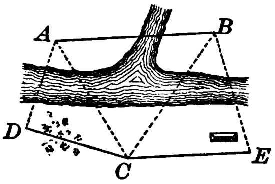

FIG. 49.

50. It is required to ascertain the length and position of an inaccessible line AB (Fig. 50), its extremities not being visible from a common point beyond the obstacles. By chaining we have $CD = 210.7$ feet, $DE = 390.4$ feet, $EF = 173.5$ feet. Then the following angles are measured: $ACD = 83^\circ\ 41'$, $CDE = 19^\circ\ 12'$ left ($180^\circ - 19^\circ\ 12'$), $CDA = 79^\circ\ 49'$, $FEB = 53^\circ\ 20'$, $DEF = 42^\circ\ 03'$ left, $EFB = 115^\circ\ 27'$.

In order to locate points suitably upon a map, find lengths AB, AD, and BE.

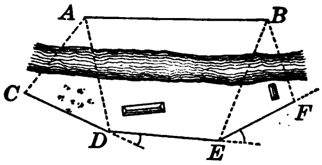

FIG. 50.

51. A tower 115 feet high casts a shadow 157 feet long upon a walk which slopes downward from its base at the rate of 1 in 10. What is the elevation of the sun above the horizon?

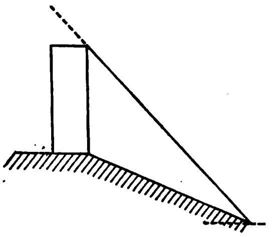

FIG. 51.

CHAPTER VII

THE GENERAL ANGLE

Only those parts of trigonometry that are necessary for the solution of triangles have been developed thus far. In this and the following chapters are considered some of the more important topics of another phase of trigonometry that is no less essential for the further study of pure and applied mathematics.

52. General definition of an angle. If a straight line rotates about one of its points, remaining always in the same plane, it generates an angle. The angle is measured by the amount of rotation by which the line is brought from its original position into its terminal position. For the small rotation leading to acute and obtuse angles this definition agrees with the customary elementary definition, the knowledge of which has been presupposed in the foregoing chapters.

As in Art. 3, counterclockwise rotation generates positive angles; clockwise rotation, negative.

In the sexagesimal system of angle measurement the standard unit is the angle produced by one complete rotation of the generating line. This angle is divided into 360 equal parts called *degrees*, the degree into 60 minutes, and the minute into 60 seconds.

In the circular system the standard unit is the *radian*, the angle produced by such a rotation that each point in the generating line describes an arc equal in length to its radius. Angular magnitudes are stated in radians and decimal fractions thereof.

Instruments are graduated and tables printed in accordance with the sexagesimal system, which is used in practical numerical calculations. Astronomers, however, employ decimal fractions of seconds, while engineers make use of tenths of minutes and decimal divisions of degrees. In theoretical discussions the radian system is commonly employed. Hereafter, in this book, the two systems will be used interchangeably.

Since the circumference of a circle is equal to 2π times its radius, where $\pi = 3.14159\cdots$, we may write the following relations between the two systems:

$$\mathbf{2\,\pi\ radians = 360^\circ}$$

$$\mathbf{1\ radian = 57.29578^\circ\cdots}$$

$$\mathbf{= 57^\circ\ 17'\ 44.8''}$$

and, in general, the number of degrees in any angle is equal to the number of radians multiplied by $\frac{180}{\pi}$, while the number of radians is equal to the number of degrees multiplied by $\frac{\pi}{180}$. Thus the straight angle is π radians; the right angle, $\frac{\pi}{2}$ radians.

If the radius of the circle is represented by r, the arc by a, and the angle, in radians, by α, we have the important relation

$$a = r\,\alpha.$$

53. Axes, quadrants, etc. Let the two axes of coördinates be assumed as in Art. 4; and, as in Art. 6, let the angle be placed upon the axis, its vertex at the origin, and its initial line extending along the X-axis toward the right. The sign and magnitude of the angle will determine the position of the terminal line, causing it to coincide with one of the axes or to fall in one of the quadrants. An angle is said to be of the first, second, third, or fourth quadrant according as its terminal line falls in that quadrant.

While the acute angle is of the first quadrant, the converse is by no means necessarily true. The terminal line of every angle, however large, must coincide with the terminal line of some positive angle less than 360° (see Fig. 52). For the purpose of trigonometry as developed in the present chapter, for every angle, positive or negative, and of any magnitude, may be substituted a positive angle less than 360°.

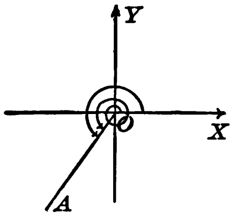

FIG. 52.

54. Definitions of the trigonometric functions. The trigonometric functions of angles of any size are defined identically as in

Art. 6. Thus for all positions of the terminal line, Fig. 53,

$$\frac{y}{v} = \sin \alpha, \quad \frac{x}{v} = \cos \alpha,$$

$$\frac{y}{x} = \tan \alpha, \quad \frac{x}{y} = \cot \alpha,$$

$$\frac{v}{x} = \sec \alpha, \quad \frac{v}{y} = \csc \alpha.$$

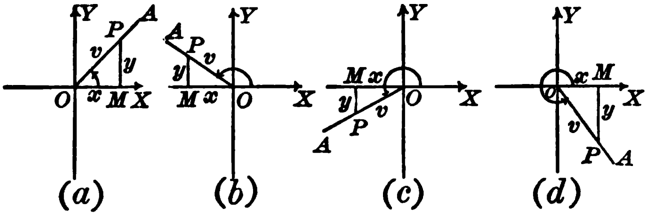

Fig. 53.

55. Signs and limitations in value. The abscissas are positive for all points in the first and fourth quadrants, negative for those in the second and third. Ordinates are positive for all points in the first and second quadrants, negative for those in the third and fourth. The radius vector is, by agreement, considered positive for all points.

In conformity with the sign law of algebra, the functions of angles of the different quadrants will have signs as displayed in the following table:

Quad.	Sine	Cosine	Tangent	Cotangent	Secant	Cosecant
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

It will be noticed that for angles of the first quadrant all six functions are positive. In each of the other quadrants one pair of mutually reciprocal functions are positive, the other two pairs are negative. These positive pairs run as follows: second quadrant, sine and cosecant: third quadrant, tangent and cotangent: fourth quadrant, cosine and secant.

The student should establish these statements regarding the signs of the functions and memorize them.

Since the lengths of the abscissa and ordinate can never exceed that of the radius vector, it follows that the sine and cosine can never be numerically greater than unity, and the secant and cosecant can never be numerically less than unity. The tangent and cotangent can have numerical values either greater or less than unity.

EXERCISE XV

1. Express in degrees, minutes, and seconds the angles $\frac{\pi^R}{4}$, $\frac{\pi^R}{3}$, $\frac{5\pi^R}{6}$, $\frac{5\pi^R}{8}$, 3^R, $\frac{3^R}{4}$.

2. Express in radians the angles 30°, 15°, 45°, 120°, 240°, 300°, 450°.

3. In a circle of radius 60 cm., what is the length of the arc which subtends at the center the angle 30°, 60°, $\frac{2\pi}{3}$, $\frac{\pi}{4}$?

4. In a circle of radius 10 inches, what is the circular measure of the angle subtended by an arc whose length is 10, 5, 20, 5π inches?

5. A friction gear consists of two tangent wheels, whose radii are 8 and 12 inches, respectively. The smaller wheel makes 4 revolutions per second. Find the number of revolutions per second made by the larger, the angular velocity of each, and the linear velocity of a point on the circumference of each. If the larger wheel is attached to the rear axle of an automobile whose rear wheel has a diameter of 30 inches, find the speed of progress of the machine.

6. The diameters of the front and rear sprocket wheels of a bicycle are 10 inches and 4 inches, respectively, and the diameter of the rear wheel is 28 inches. Find the rate of pedaling when the bicycle is traveling 12 miles per hour, the corresponding angular velocities of the two sprocket wheels, and the linear velocity of the chain.

7. Determine the quadrant to which each of the following angles belongs: 210°, 465°, 745°, $-830°$, $\frac{10\pi}{3}$, $\frac{15\pi}{4}$, $-\frac{8\pi}{3}$.

8. Determine the signs of the functions of the following angles: 240°, 330°, 400°, $\frac{5\pi}{3}$, $-\frac{7\pi}{4}$, 6^R.

9. Show that the quadrant to which an angle belongs is determined if the signs of any two non-reciprocal functions are given.

10. To what quadrant does an angle belong if its sine and tangent are negative; its secant and cotangent positive; sine and secant negative; tangent and cosine positive?

11. Determine the quadrants of the following angles:

$$\sin^{-1}\tfrac{2}{3};\ \arccos -\tfrac{5}{13};\ \arctan\tfrac{3}{4};\ \cot^{-1} -\tfrac{7}{12}.$$

12. Determine the quadrants of the following angles:

$$\sin^{-1}\tfrac{3}{5} \equiv \cot^{-1} -\tfrac{4}{3};\ \arccos -\tfrac{5}{13} \equiv \operatorname{arccsc}\tfrac{13}{12}.$$

13. For what values of α is $\sin\alpha - \cos\alpha$ positive?

14. For what values of α is $\tan\alpha - \cot\alpha$ negative?

15–20. Find the missing values in the following table:

∠	sin	cos	tan	cot	sec	csc	QUAD.
α	$\frac{20}{29}$						II
β		$-\frac{28}{53}$					III
γ			$-\frac{9}{40}$				III
δ				$-\frac{15}{8}$			IV
θ					$\frac{25}{7}$		IV
ϕ						$-\frac{13}{5}$	III

56. Variation of the trigonometric functions. A change in the angle will produce a corresponding change in the values of the coördinates and in their ratios. If, for convenience, the chosen point in the terminal line of the angle is maintained at a constant distance from the vertex, the radius vector will retain the constant value $+v$.

As the angle θ increases continuously from 0° to 360°, the abscissa and ordinate vary continuously between the limits $-v$ and $+v$. As θ increases from 0° to 90°, x is positive and decreases from v to 0; as θ increases from 90° to 180°, x is negative and decreases (algebraically) from 0 to $-v$; as θ increases from 180° to 270°, x is negative and increases from $-v$ to 0; and as θ increases from 270° to 360°, x is positive and increases from 0 to v. As θ increases from 0° to 90°, y is positive and increases from 0 to v; as θ increases from 90° to 180°, y is positive and decreases from v to 0; as θ increases from 180° to 270°, y is negative and decreases from 0 to $-v$; as θ increases from 270° to 360°, y is negative and increases from $-v$ to 0. Upon introducing these varying values into the ratio definitions, we are enabled to trace the variation of the trigonometric functions.

We see, for example, that as θ increases from 0° to 360°, $\tan\theta$ continually increases algebraically, changing sign from negative to positive through the value 0 as θ passes through 0°, 180°, and 360°, and from positive to negative by becoming infinite as θ passes through 90° and 270°. There is an infinite discontinuity in $\tan \theta$, for $\theta = 90°$ and $\theta = 270°$.

QUERY. Which of the trigonometric functions other than the tangent become infinite and therefore discontinuous?

The student should trace the variation of each function in detail, stating the narrative verbally.

57. Graphs of the trigonometric functions. The whole behavior of each function can be conveniently represented by means of the graphical method already introduced in Art. 4. Assume a pair of axes of coördinates, as in Art. 4, and along the X-axis to the right lay off equal spaces corresponding to the number of degrees in the angle θ. At each point in the X-axis erect a perpendicular whose length is proportional to the value of the sine of that angle. Each point thus determined has the property that its abscissa represents the angle θ and its ordinate the corresponding value of $\sin \theta$. Now having located a sufficient number of points, draw through them a smooth curve. It will be seen that the value, sign, and variation of the sign at each instant is fully exhibited by the ordinate, position, and inclination of the curve or graph. The same may be done for each of the functions.

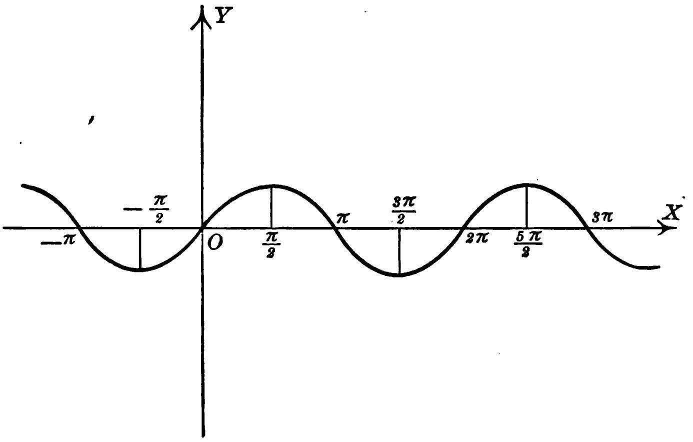

FIG. 54. Graph of $\sin \theta$.

The graphs of the different functions are here presented. The student should trace carefully the intimate and exact correspondence of the graphical and the verbal narratives.

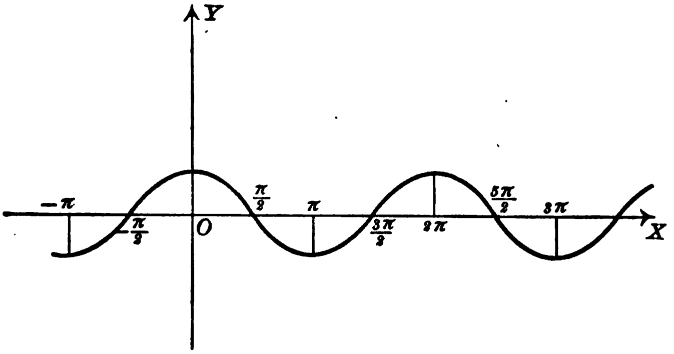

Fig. 55. Graph of cos θ.

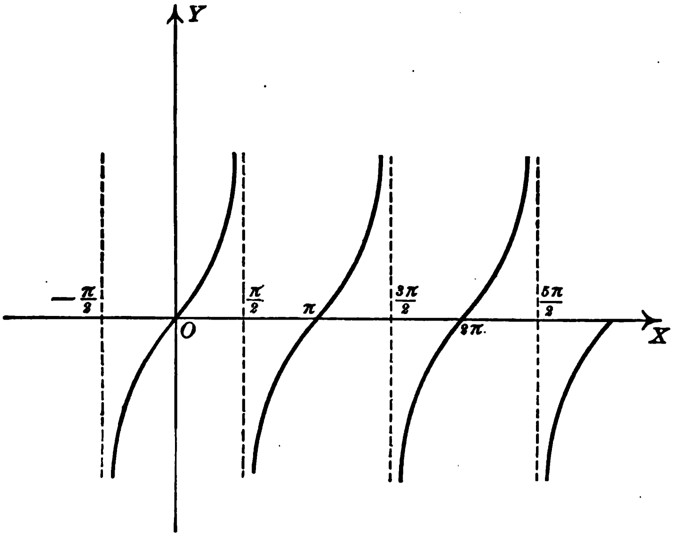

Fig. 56. Graph of tan θ.

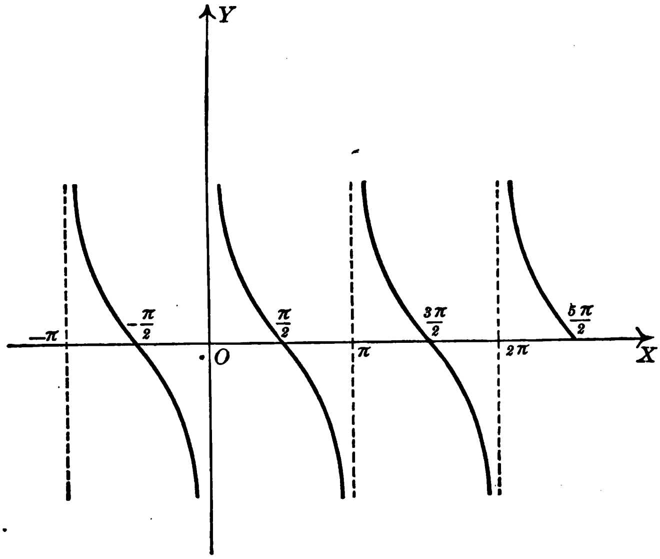

FIG. 57. Graph of cot θ.

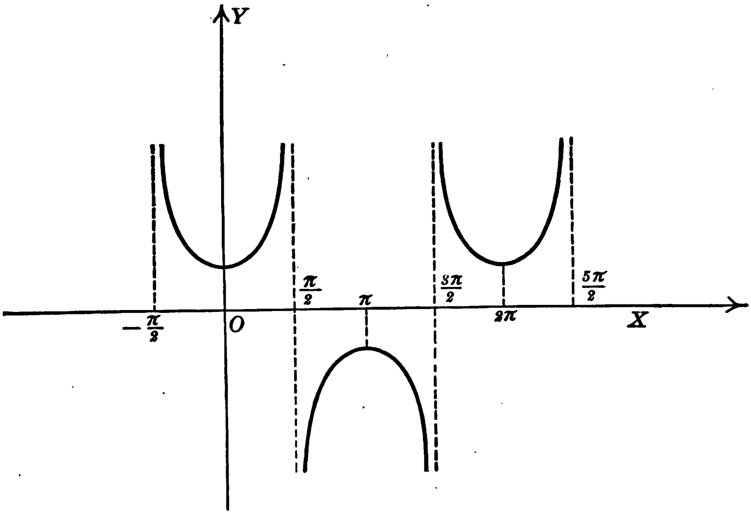

FIG. 58. Graph of sec θ.

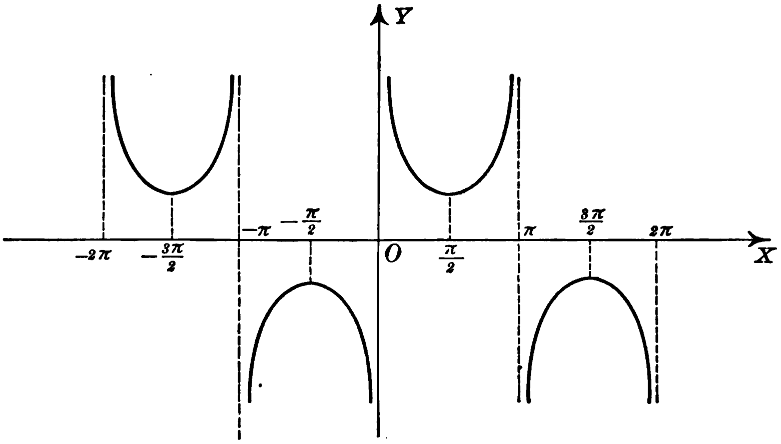

FIG. 59. Graph of csc θ.

58. Functions of 270° and 360°. By the method of limits employed in Art. 12, we get the following sets of values:

$$\sin 270^\circ = -1, \qquad \cos 270^\circ = 0,$$
$$\tan 270^\circ = \infty, \qquad \cot 270^\circ = 0,$$
$$\sec 270^\circ = \infty. \qquad \csc 270^\circ = -1.$$
$$\sin 360^\circ = 0, \qquad \cos 360^\circ = 1,$$
$$\tan 360^\circ = 0, \qquad \cot 360^\circ = \infty,$$
$$\sec 360^\circ = 1, \qquad \csc 360^\circ = \infty.$$

Here ∞ is used as before to denote the value of a fraction whose numerator remains finite while its denominator approaches zero. The sign + or − is prefixed to the symbol ∞ according as the variable becomes ∞ through a positive or a negative sequence of values. In the light of this discussion the values of the functions of $k \times \frac{\pi}{2}$ (k any integer) may be tabulated, the upper of the pair of double signs arising when the angle approaches the critical value from below.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0	∓ 0	$+1$	∓ 0	$\mp\infty$	$+1$	$\mp\infty$
$\frac{\pi}{2}$	$+1$	± 0	$\pm\infty$	± 0	$\pm\infty$	$+1$
π	± 0	-1	∓ 0	$\mp\infty$	-1	$\pm\infty$
$\frac{3}{2}\pi$	-1	∓ 0	$\pm\infty$	± 0	$\mp\infty$	-1
2π	± 0	$+1$	∓ 0	$\mp\infty$	$+1$	$\mp\infty$

EXERCISE XVI

1. Trace the variation, as θ varies, (*a*) of $\sin 2\theta$; (*b*) of $\cot \frac{\theta}{2}$.

2. Trace the variation, as θ varies, (*a*) of $\tan 2\theta$; (*b*) of $\cos \frac{\theta}{2}$.

3. Draw the graph of $\cos 2\theta$.

4. Draw the graph of $\sin 3\theta$.

5. In what points will a horizontal line $\frac{1}{2}$ unit above the X-axis intersect the graph of $\sin \theta$? Explain the significance of the result.

6. In what points will a horizontal line 1 unit above the X-axis intersect the graph of $\tan \theta$? Explain.

7. If the graphs of $\tan \theta$ and $\cot \theta$ are drawn on the same axes to the same scale, where will they intersect? What is the significance?

8. If the graphs of $\sin \theta$ and $\cos \theta$ are drawn on the same axes to the same scale, where will they intersect? What is the significance?

9. Construct the graph of $\log_{10} x$, taking values of the number x as abscissas and the corresponding logarithms as ordinates.

59. Fundamental relations. Just as in Art. 9 we find, by inspection,

$$\csc \alpha = \frac{1}{\sin \alpha}, \tag{1}$$

$$\sec \alpha = \frac{1}{\cos \alpha}, \tag{2}$$

$$\cot \alpha = \frac{1}{\tan \alpha}; \tag{3}$$

by division,

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}, \tag{4}$$

$$\cot \alpha = \frac{\cos \alpha}{\sin \alpha}; \tag{5}$$

by virtue of the Pythagorean proposition,

$$\sin^2 \alpha + \cos^2 \alpha = 1, \tag{6}$$

$$\tan^2 \alpha + 1 = \sec^2 \alpha, \tag{7}$$

$$\cot^2 \alpha + 1 = \csc^2 \alpha. \tag{8}$$

The student should prove that all these formulas conform, for angles in all quadrants, to the algebraic law of signs.

60. Line representations of the trigonometric functions. As the names tangent and secant indicate, the trigonometric functions were originally defined as certain lines measured in terms of a standard unit line. The adoption of the abstract ratios, as in this book, is of comparatively recent date. It is both interesting and advantageous to know the line representations and show that they lead to the same science of trigonometry as do the ratio definitions.

The line representations most frequently used involve the use of a unit circle, *i.e.* a circle of radius unity. It is evident that we may replace each of the defining ratios of Art. 54 by an equal ratio so chosen that its denominator is positive unity. The value of the ratio will be equal to that of the numerator. In other words, if a positive unit radius is taken as the denominator, the length and sign of the numerator will represent the function in magnitude and sign. We have, then, simply to select six lines whose ratios to the radius agree with the definitions of Art. 54. The ratio of the subtended arc to the radius is, by Art. 52, the circular measure of the angle.

Suppose, then, a circle of unit radius drawn with its center at the origin of coördinates.

The angle is placed upon the axes just as in Art. 6, and from the point P of intersection of the terminal line with the circle, perpendiculars MP and NP are drawn to the two axes. From the two points A and B where the positive axes cut the circle, tangents AT and BS are drawn meeting the terminal line (produced if necessary) in the points T and S.

Since P, T, S, Figs. 60–63, lie in the terminal line, we have, at once, in accordance with Art. 54 (or Art. 6):

$$\sin\alpha = \frac{MP}{OP}, \qquad \cos\alpha = \frac{NP}{OP},$$

$$\tan\alpha = \frac{AT}{OA}, \qquad \cot\alpha = \frac{BS}{OB},$$

$$\sec\alpha = \frac{OT}{OA}, \qquad \csc\alpha = \frac{OS}{OB}.$$

But by construction,

$$OP = OA = OB = 1.$$

These denominators may then be suppressed and the functions represented graphically as indicated below:

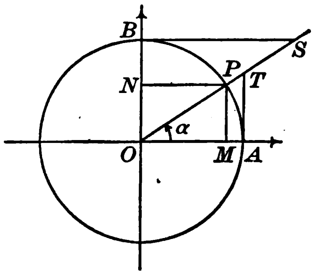

FIG. 60.

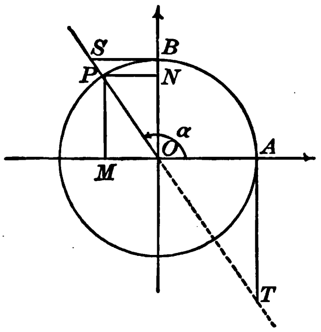

FIG. 61.

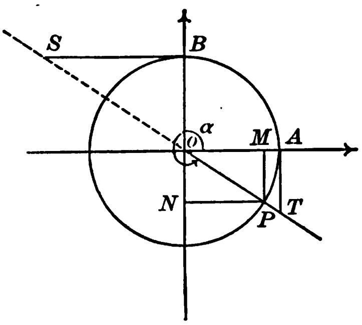

FIG. 62.

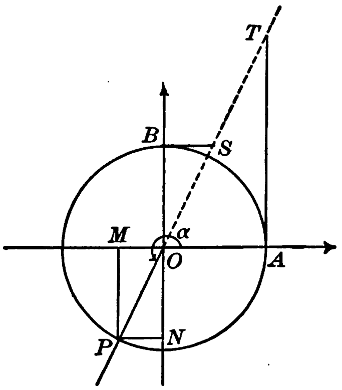

FIG. 63.

$$\sin\alpha = MP,\qquad \cos\alpha = NP,$$
$$\tan\alpha = AT,\qquad \cot\alpha = BS,$$
$$\sec\alpha = OT,\qquad \csc\alpha = OS.$$

Moreover, the angle, in radians, is represented as follows:

$$\alpha = \frac{\text{arc } AP}{OP} = \text{arc } AP.$$

According to the modern view, the line is not the function, but by its length and direction represents the function in magnitude and sign.

Note that the line representing the tangent is always drawn from the point A and that representing the cotangent from B. All the lines are read *from* the axes *to* the terminal line. Horizontal lines are positive toward the right, negative toward the left. Vertical lines are positive upward, negative downward.

By means of the Pythagorean proposition, and the theorems concerning similar triangles, the fundamental relations given in the preceding article, as also the limitations of value stated in Art. 55, are readily established. So, also, the subsequent theorems of trigonometry may be interpreted by means of the line representation of the trigonometric functions. This graphic interpretation frequently presents special advantages. This is the case, for example, in the investigation of the variation of the functions considered in Art. 56. So, too, the construction of the graphs of the functions as treated in Art. 57 is facilitated, since the lengths of the defining lines may be transferred by the use of dividers.

EXERCISE XVII

Find the values of the following expressions:

1. $\cos^4 \alpha - \sin^4 \alpha$, when $\alpha = \arctan(-\frac{1}{2})$, in the 2d quadrant.
2. $\frac{\cos \alpha}{1 - \tan \alpha} + \frac{\sin \alpha}{1 - \cot \alpha}$, when $\alpha = \sec^{-1}(-3)$, in the 3d quadrant.
3. $\frac{\tan \alpha + \sec \alpha - 1}{\tan \alpha - \sec \alpha + 1}$, when $\alpha = \arcsin(-\frac{11}{61})$, in the 4th quadrant.
4. $\frac{1}{\csc \alpha - \cot \alpha} + \frac{1}{\csc \alpha + \cot \alpha}$, when $\alpha = \cos^{-1}\frac{20}{29}$, in the 4th quadrant.

Solve the following equations, finding all the angles less than 2π that satisfy each equation:

5. $\cos \beta = \frac{1}{2}$.
6. $\tan \beta = -\sqrt{3}$.
7. $\sin 2\alpha = -\frac{1}{2}\sqrt{3}$.
8. $\cot 3\alpha = 1$.
9. $4 \sin^2 \alpha - 4 \cos \alpha - 1 = 0$.
10. $3 \tan^2 \beta - 1 = 0$.
11. $2 \sin \beta \cos \beta - \sin \beta = 0$.
12. $2 \sin \alpha + \sqrt{3} \tan \alpha = 0$.

In exercises 13–24, verify the given identities by transforming the first member into the second.

13. $(\sin \alpha + \cos \alpha)(\cot \alpha + \tan \alpha) = \sec \alpha + \csc \alpha$.
14. $(\sec \alpha - \cos \alpha)(\csc \alpha - \sin \alpha) = \sin \alpha \cos \alpha$.
15. $\frac{\tan \alpha + \cot \beta}{\cot \alpha + \tan \beta} = \tan \alpha \cot \beta$.

16. $(r\cos\theta)^2+(r\sin\theta\cos\phi)^2+(r\sin\theta\sin\phi)^2=r^2.$

17. $\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\cdot\dfrac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}=1.$

18. $\csc\alpha(\sec\alpha-1)-\cot\alpha(1-\cos\alpha)=\tan\alpha-\sin\alpha.$

19. $(\sin\alpha\cos\beta-\cos\alpha\sin\beta)^2+(\cos\alpha\cos\beta+\sin\alpha\sin\beta)^2=1.$

20. $\sec\alpha\csc\alpha(1-2\cos^2\alpha)+\cot\alpha=\tan\alpha.$

21. $(\sin\alpha\cos\beta+\cos\alpha\sin\beta)^2+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)^2=1.$

22. $\sec^2\alpha\csc^2\alpha-\dfrac{(1-\tan^2\alpha)^2}{\tan^2\alpha}=4.$

23. $(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\alpha-\sqrt{-1}\sin\alpha)=1.$

24. $(\cos\alpha+\sqrt{-1}\sin\alpha)^2+(\cos\alpha-\sqrt{-1}\sin\alpha)^2=4\cos^2\alpha-2.$

25. By means of Fig. 60 show that, when θ is acute and measured in radians, $\sec\theta>\tan\theta>\theta>\sin\theta.$

26. By means of Fig. 60 show that, when θ is acute and measured in radians, $\csc\theta>\cot\theta>\left(\frac{\pi}{2}-\theta\right)>\cos\theta.$

61. Periodicity of the trigonometric functions. It was pointed out, in Art. 53, that if two angles differing by an integral multiple of 360° are placed on the axes, their terminal lines coincide. As an immediate consequence, it follows that corresponding functions of the two angles are identical. Thus we may write

$$\sin(2k\pi+\alpha)=\sin\alpha,$$

and, in general,

$$F(2k\pi+\alpha)=F(\alpha),$$

where F denotes the same function in both members of the equation, and k is an integer.

62. Functions of $\left[k\cdot\frac{\pi}{2}\pm\alpha\right]$. Precisely as in Art. 10, 37, and 38, we may express the functions of the angles $\pm\alpha$, $90°\pm\alpha$, $180°\pm\alpha$, $270°\pm\alpha$, $360°\pm\alpha$, and other similarly compounded angles in terms of the functions of α, no matter what the quadrant of the angle α. Because of the periodicity brought out in the preceding article, it is not necessary to carry the investigation beyond the five multiples of the right angle mentioned; indeed, the fifth reduces to the first. On account of the double signs and the possibility of α belonging to any one of the four quadrants, there exist thirty-two distinct cases. The demonstration is the same for all cases, involving the same proportionality of sides of similar triangles and the same question of agreement or opposition of signs. The working out of the proof in three characteristic instances should be sufficient to enable the student to do the same for any and all cases. The theorem is, however, somewhat elusive, and the student can completely master it and render it an infallible instrument only by actual careful construction and proof of most of the cases. Upon first study it may be well to limit consideration to the cases in which α is of the first quadrant.

Let it be required first to express the functions of $(180° + \alpha)$ in terms of functions of α, when α is an angle of the first quadrant. If, in Fig. 64, $\angle XOA = \alpha$, then $\angle XOB = \beta = 180° + \alpha$. The two triangles OMP and ONQ are similar, the pairs of corresponding sides being v and v', x and x', and y and y'. Notice also that x' and y' are negative, all the other sides being positive. Giving due attention to signs, we may write:

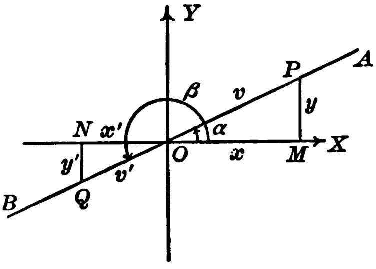

Fig 64.

$$\sin(180° + \alpha) \equiv \sin\beta = \frac{y'}{v'} = -\frac{y}{v} = -\sin\alpha,$$

$$\cos(180° + \alpha) \equiv \cos\beta = \frac{x'}{v'} = -\frac{x}{v} = -\cos\alpha,$$

$$\tan(180° + \alpha) \equiv \tan\beta = \frac{y'}{x'} = \frac{y}{x} = \tan\alpha,$$

$$\cot(180° + \alpha) \equiv \cot\beta = \frac{x'}{y'} = \frac{x}{y} = \cot\alpha,$$

$$\sec(180° + \alpha) \equiv \sec\beta = \frac{v'}{x'} = -\frac{v}{x} = -\sec\alpha,$$

$$\csc(180° + \alpha) \equiv \csc\beta = \frac{v'}{y'} = -\frac{v}{y} = -\csc\alpha.$$

Again, let it be required to express the functions of $(270° - \alpha)$ in terms of functions of α, when α is of the first quadrant. In Fig. 65, $\angle XOA = \alpha$, $\angle XOB = \beta = 270° - \alpha$. The two triangles OMP and ONQ are similar, the pairs of corresponding sides now being v and v', x and y', and y and x'. The sides x' and y' are negative, all the others positive. We may then write:

$$\sin(270^\circ-\alpha)\equiv\sin\beta=\frac{y'}{v'}=-\frac{x}{v}=-\cos\alpha.$$

$$\cos(270^\circ-\alpha)\equiv\cos\beta=\frac{x'}{v'}=-\frac{y}{v}=-\sin\alpha.$$

$$\tan(270^\circ-\alpha)\equiv\tan\beta=\frac{y'}{x'}=\frac{x}{y}=\cot\alpha.$$

$$\cot(270^\circ-\alpha)\equiv\cot\beta=\frac{x'}{y'}=\frac{y}{x}=\tan\alpha.$$

$$\sec(270^\circ-\alpha)\equiv\sec\beta=\frac{v'}{x'}=-\frac{v}{y}=-\csc\alpha.$$

$$\csc(270^\circ-\alpha)\equiv\csc\beta=\frac{v'}{y'}=-\frac{v}{x}=-\sec\alpha.$$

As a third and especially important instance, let us find the functions of $-\alpha$, when α is of the second quadrant. In Fig. 66, $XOA=\alpha$, $XOB=\beta=-\alpha$. The two triangles OMP and ONQ are similar, the pairs of corresponding sides being v and v', x and x', y and y', while x, x', and y' are negative.

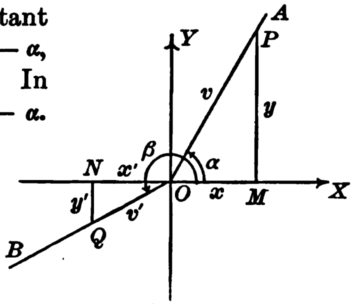

FIG. 65.

We then have, as before:

$$\sin(-\alpha)\equiv\sin\beta=\frac{y'}{v'}=-\frac{y}{v}=-\sin\alpha,$$

$$\cos(-\alpha)\equiv\cos\beta=\frac{x'}{v'}=\frac{x}{v}=\cos\alpha,$$

$$\tan(-\alpha)\equiv\tan\beta=\frac{y'}{x'}=\frac{y}{x}=-\tan\alpha.$$

$$\cot(-\alpha)\equiv\cot\beta=\frac{x'}{y'}=-\frac{x}{y}=-\cot\alpha,$$

$$\sec(-\alpha)\equiv\sec\beta=\frac{v'}{x'}=\frac{v}{x}=\sec\alpha,$$

$$\csc(-\alpha)\equiv\csc\beta=\frac{v'}{y'}=-\frac{v}{x}=-\csc\alpha.$$

It will be noticed that whenever the number of right angles involved is even the pairs of corresponding sides are v and v', x and x', y and y'; while whenever the number of right angles is odd the pairs of corresponding sides are v and v', x *and* y', y *and* x'. Thus we have the theorem: *Any function of an even number of right angles plus or minus α is numerically equal to the same function of α; any function of an odd number of right angles plus or minus α is numerically equal to the corresponding co-function of α; the agreement or opposition of signs is to be determined from the quadrants of α and of the compound angle.* It may easily be verified that in all cases this agreement or opposition of signs is the same as when α is of first quadrant.

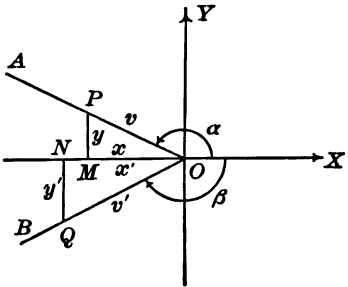

FIG. 66.

The general theorem may also be stated as follows: *If the sum or difference of two angles is an even number of right angles, the functions of the one are numerically equal to the same functions of the other. If the sum or difference of two angles is an odd number of right angles, the functions of the one are numerically equal to the corresponding co-functions of the other. The agreement or opposition of signs is to be determined from the quadrants of the two angles.*

The significance of the theorem is made clear by application to an example: Required to find the value of $\cos(810° + \alpha)$. Here $810° = 9 \times 90°$, an odd number of right angles. When α is considered as of the first quadrant (and its functions consequently positive), the compound angle $(810° + \alpha)$ is of the second quadrant and hence its cosine is negative. The required relation is, therefore,

$$\cos(810° + \alpha) = -\sin\alpha,$$

which holds for all values of α.

Again, to find the value of tan 1230°. We have

$$1230° = 14 \times 90° - 30°, \text{ and is of second quadrant.}$$

Then $$\tan 1230° = -\tan 30° = -\frac{1}{\sqrt{3}}.$$

The student may, if he prefers, construct the figure and proceed as in the demonstration just given.

As a consequence of these relations, it follows that to every inverse function correspond two angles, lying between 0 and 2π.

Thus $\quad \text{arc}\sin a=\alpha$ and $\pi-\alpha$,

$\text{arc}\cos b=\alpha$ and $2\pi-\alpha$,

$\text{arc}\tan c=\alpha$ and $\pi+\alpha$,

$\text{arc}\cot d=\alpha$ and $\pi+\alpha$,

$\text{arc}\sec e=\alpha$ and $2\pi-\alpha$,

$\text{arc}\csc f=\alpha$ and $\pi-\alpha$.

These statements should be verified by the student.

EXERCISE XVIII

Express in terms of a positive angle less than 45°:

1. $\sin 700^\circ$.
2. $\cos 260^\circ$.
3. $\tan 436^\circ$.
4. $\cot -35^\circ$.
5. $\csc 930^\circ$.
6. $\sec 1400^\circ$.

Find the value of $\cos\alpha+\sin\alpha$ and of $\tan\alpha-\cot\alpha$ when α has the value

7. $\frac{\pi}{6}$.
8. $-\frac{2\pi}{3}$.
9. $\frac{19\pi}{4}$.
10. $\frac{7\pi}{6}$.
11. $\frac{11\pi}{3}$.
12. $-\frac{9\pi}{4}$.

Find all the values between 0° and 360° of

13. $\text{arctan}\sqrt{3}$.
14. $\csc^{-1}(-\sqrt{2})$.
15. $\text{arccos}(-.5)$.
16. $\text{arcsec}\ 2$.
17. $\text{arccot}(-1)$.
18. $\sin^{-1}(-\frac{1}{2}\sqrt{3})$.

Find the value of

19. $\sin 480^\circ\sin 690^\circ+\cos(-420^\circ)\cos 600^\circ$.
20. $\tan 840^\circ\cot 420^\circ+\tan(-300^\circ)\cot(-120^\circ)$.
21. $\tan\frac{17\pi}{6}\tan\frac{14\pi}{3}+\cot\left(-\frac{11\pi}{6}\right)\cot\left(-\frac{4\pi}{3}\right)$.
22. $\sin\frac{19\pi}{6}\cos\left(-\frac{11\pi}{6}\right)-\sin\frac{7\pi}{3}\cos\left(-\frac{4\pi}{3}\right)$.

23. If $\sin 200° \, 30' = -.35$, find $\cos 830° \, 30'$.

24. If $\tan 558° \, 26' = \frac{1}{3}$, find $\cot 468° \, 26'$.

25. If $\cot 520° = -a$, find $\sin 160°$.

26. If $\cos 590° = -m$, find $\tan 850°$.

27. Express $\cos (\alpha - 90°)$ as a function of α.

28. Express $\sin (\alpha - 180°)$ as a function of α.

29. Express $\tan (\alpha - 360°)$ as a function of α.

30. Express $\cot (\alpha - 270)°$ as a function of α.

CHAPTER VIII

FUNCTIONS OF TWO ANGLES

63. Formulas for sin $(\alpha+\beta)$ and cos $(\alpha+\beta)$. Suppose α and β to be acute angles. In Fig. 67 $(\alpha+\beta)$ is acute; in Fig. 68 $(\alpha+\beta)$ is obtuse. The following demonstration applies to both figures.

Let $\angle XOA=\alpha$, $\angle AOB=\beta$; then $\angle XOB=\alpha+\beta$. From P, a point in OB, draw PM perpendicular to OX, PQ perpendicular to OA, and from Q draw QN perpendicular to OX, and QR perpendicular to MP. The angle $RPQ=\alpha$ and $RP=QP\cos\alpha$, $RQ=QP\sin\alpha$, by Art. 16. By the same article,

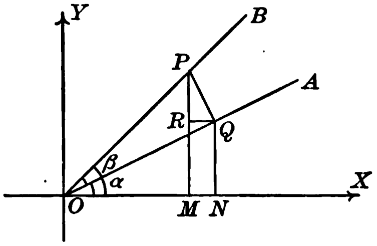

FIG. 67.

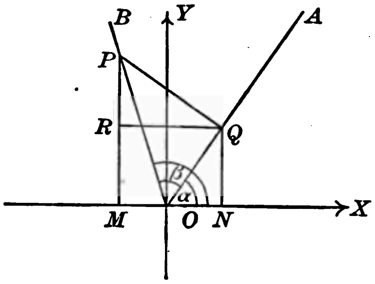

FIG. 68.

$$MP=OP\sin(\alpha+\beta).$$

Also

$$\begin{aligned}MP&=MR+RP=NQ+RP\\&=OQ\sin\alpha+QP\cos\alpha\\&=OP\sin\alpha\cos\beta+OP\cos\alpha\sin\beta.\end{aligned}$$

Equating the two values of MP and dividing through by the common factor OP, we have the theorem

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta. \quad (1)$$

In like manner

$$OM = OP \cos(\alpha + \beta),$$

and also

$$\begin{aligned} OM &= ON - MN = ON - RQ \\ &= OQ \cos\alpha - QP \sin\alpha \\ &= OP \cos\alpha \cos\beta - OP \sin\alpha \sin\beta. \end{aligned}$$

Hence the companion theorem

$$\boldsymbol{\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta.} \qquad (2)$$

These are called the *addition formulas* and are fundamental in trigonometry.

64. Extension of addition formulas. The two formulas of the last article were proved only for angles both of the first quadrant. It remains to be shown that they hold when α and β denote any angles.

First, let α be an angle of the second quadrant. Then ϕ $(= \alpha - 90°)$ is an angle of the first quadrant. Now $\alpha = 90° + \phi$, so that, by Art. 38,

$$\sin\alpha = \cos\phi, \quad \cos\alpha = -\sin\phi.$$

Since ϕ is of the first quadrant, the formulas of Art. 63 apply and we have

$$\begin{aligned} \sin(\alpha + \beta) &= \sin(90° + \overline{\phi + \beta}) \\ &= \cos(\phi + \beta) \\ &= \cos\phi \cos\beta - \sin\phi \sin\beta, \\ &= \sin\alpha \cos\beta + \cos\alpha \sin\beta. \end{aligned}$$

Likewise

$$\begin{aligned} \cos(\alpha + \beta) &= \cos(90° + \overline{\phi + \beta}) \\ &= -\sin(\phi + \beta) \\ &= -\sin\phi \cos\beta - \cos\phi \sin\beta, \\ &= \cos\alpha \cos\beta - \sin\alpha \sin\beta. \end{aligned}$$

The formulas are therefore true when one angle is of the first and the other of the second quadrant. By adding 90° successively to each of the angles, the formulas are established for two positive angles of all quadrants. If one of the angles is negative, it can be augmented by such an integral multiple of 360° as to produce a positive angle possessing the same functions.

The addition formulas are, therefore, true for angles of any size.

EXERCISE XIX

Evaluate the addition formulas for

1. $\alpha = 60°$, $\beta = 30°$.
2. $\alpha = 45°$, $\beta = 90°$.
3. $\alpha = 240°$, $\beta = 150°$.
4. $\alpha = 300°$, $\beta = 150°$.
5. $\alpha = \arctan \frac{3}{4}$, $\beta = \arccos\left(-\frac{8}{17}\right)$, α first quadrant, β second.
6. $a = \sin^{-1}\left(-\frac{5}{13}\right)$, $\beta = \cot^{-1} \frac{7}{24}$, α fourth quadrant, β third.

Find the value of

7. $\cos\left(\frac{\pi}{4} + \alpha\right) \cos\left(\frac{\pi}{4} + \beta\right) - \sin\left(\frac{\pi}{4} + \alpha\right) \sin\left(\frac{\pi}{4} + \beta\right)$.
8. $\sin\left(\frac{\pi}{3} + \alpha\right) \cos\left(\frac{\pi}{6} + \beta\right) + \cos\left(\frac{\pi}{3} + \alpha\right) \sin\left(\frac{\pi}{6} + \beta\right)$.
9. $\sin(1 + n)\alpha \cos(1 - n)\alpha + \cos(1 + n)\alpha \sin(1 - n)\alpha$.
10. $\cos(1 + n)\alpha \cos(1 - n)\alpha - \sin(1 + n)\alpha \sin(1 - n)\alpha$.
11. $\sin(\theta + \phi)\cos(\theta - \phi) + \cos(\theta + \phi)\sin(\theta - \phi)$.
12. $\cos(\theta - \phi)\cos\phi - \sin(\theta - \phi)\sin\phi$.
13. Evaluate the addition formulas for $\alpha = 60°$, $\beta = 45°$, and thus find $\sin 105°$, $\cos 105°$, $\sin 15°$, $\cos 15°$.
14. Evaluate the addition formulas for $\alpha = 45°$, $\beta = 30°$, and thus find $\sin 75°$, $\cos 75°$, $\sin 15°$, $\cos 15°$.

65. Subtraction formulas. In the addition formulas replace β by $-\beta$. We have

$$\sin(\alpha - \beta) = \sin\alpha \cos(-\beta) + \cos\alpha \sin(-\beta).$$

But by Art. 62,

$$\sin(-\beta) = -\sin\beta, \quad \cos(-\beta) = \cos\beta.$$

Making this substitution, we have

$$\boldsymbol{\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta.} \tag{1}$$

In like manner

$$\cos(\alpha - \beta) = \cos\alpha \cos(-\beta) - \sin\alpha \sin(-\beta),$$

or, by the same substitution,

$$\boldsymbol{\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta.} \tag{2}$$

66. Formulas for $\tan(\alpha \pm \beta)$, $\cot(\alpha \pm \beta)$. From Arts. 59 and 63 we have

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$

$$=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

$$=\frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

or, finally,

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}. \qquad (1)$$

In like manner we may derive

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}. \qquad (2)$$

Again,

$$\cot(\alpha+\beta)=\frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)},$$

$$=\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta},$$

$$=\frac{\dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}-\dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta}+\dfrac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}},$$

or

$$\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}. \qquad (3)$$

Likewise

$$\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}. \qquad (4)$$

EXERCISE XX

1. Demonstrate geometrically the formula for $\sin(\alpha-\beta)$, when $\alpha > \beta$, both acute.

2. Demonstrate geometrically the formula for $\cos(\alpha-\beta)$, when $\alpha > \beta$, both acute.

Evaluate the formulas of Art. 65 for

3. $\alpha = 60°, \beta = 120°$.

4. $\alpha = 240°, \beta = 150°$.

5. $\alpha = \arcsin \frac{8}{17}, \beta = \arctan - \frac{16}{63}$.

6. $\alpha = \cot^{-1} \frac{21}{20}, \beta = \cos^{-1} \frac{33}{65}$.

Evaluate the formulas of Art. 66 for

7. $\alpha = 330°, \beta = 150°$.

8. $\alpha = 210°, \beta = 300°$.

9. $\alpha = \cos^{-1} \frac{5}{13}, \beta = \tan^{-1} (-\frac{3}{4})$.

10. $\alpha = \operatorname{arccot} \frac{60}{11}, \beta = \arctan \frac{60}{11}$.

11. Find the functions of 15° by putting $\alpha = 45°, \beta = 30°$.

12. Find the functions of 15° by putting $\alpha = 60°, \beta = 45°$.

Show that

13. $\sin (\alpha + \beta) \sin (\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta = \cos^2 \beta - \cos^2 \alpha$.

14. $\cos (\alpha + \beta) \cos (\alpha - \beta) = \cos^2 \alpha - \sin^2 \beta = \cos^2 \beta - \sin^2 \alpha$.

Expand by successive applications of the formulas:

15. $\sin (\alpha + \beta + \gamma)$.

16. $\cos (\alpha + \beta + \gamma)$.

17. $\tan (\alpha + \beta + \gamma)$.

18. $\cot (\alpha + \beta + \gamma)$.

Show that

19. $\sin\left(\frac{\pi}{3} + \alpha\right) - \sin\left(\frac{\pi}{3} - \alpha\right) = \sin \alpha$.

20. $\cos\left(\frac{\pi}{6} + \alpha\right) + \cos\left(\frac{\pi}{6} - \alpha\right) = \sqrt{3} \cos \alpha$.

67. Functions of twice an angle. In the addition formulas of Arts. 63 and 66, place $\beta = \alpha$. We then obtain

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha, \tag{1}$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha, \tag{2}$$

$$= 1 - 2 \sin^2 \alpha, \tag{2a}$$

$$= 2 \cos^2 \alpha - 1. \tag{2b}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}. \tag{3}$$

$$\cot 2\alpha = \frac{\cot^2 \alpha - 1}{2 \cot \alpha}. \tag{4}$$

68. Functions of half an angle. From Art. 67 we may write

$$\cos 2\beta = 1 - 2 \sin^2 \beta,$$

and solving for $\sin\beta$,

$$\sin\beta = \sqrt{\tfrac{1}{2}(1-\cos 2\beta)}.$$

Now placing $2\beta = \alpha$, so that $\beta = \frac{\alpha}{2}$, we obtain

$$\sin\tfrac{1}{2}\alpha = \sqrt{\tfrac{1}{2}(1-\cos\alpha)}. \tag{1}$$

Similarly

$$\cos 2\beta = 2\cos^2\beta - 1;$$

so that

$$\cos\beta = \sqrt{\tfrac{1}{2}(1+\cos 2\beta)},$$

and, with the same substitution,

$$\cos\tfrac{1}{2}\alpha = \sqrt{\tfrac{1}{2}(1+\cos\alpha)}. \tag{2}$$

Dividing the first formula by the second, we get

$$\tan\frac{1}{2}\alpha = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}, \tag{3}$$

and inverting,

$$\cot\frac{1}{2}\alpha = \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}. \tag{4}$$

Rationalizing the numerators of the last two formulas, we get other useful forms,

$$\tan\frac{1}{2}\alpha = \frac{1-\cos\alpha}{\sin\alpha}, \tag{5}$$

$$\cot\frac{1}{2}\alpha = \frac{1+\cos\alpha}{\sin\alpha}. \tag{6}$$

QUERY. — Why are formulas (1) to (4) ambiguous in sign, while (5) and (6) are apparently not?

EXERCISE XXI

Find the values of

1. The functions of 60° from those of 30°.
2. The functions of 120° from those of 60°.
3. The functions of 75° from those of 150°.
4. The functions of 15° from those of 30°.

Find the values of the functions of

5. $2\arctan\frac{9}{40}$.
6. $2\cos^{-1}(-\frac{5}{13})$.
7. $\frac{1}{2}\sin^{-1}(-\frac{24}{25})$.
8. $\frac{1}{2}\arctan\frac{119}{120}$.
9. $\arcsin\frac{5}{13} + 2\,\text{arccot}\,\frac{4}{3}$.
10. $\arctan\frac{5}{12} - 2\arccos\frac{3}{5}$.

Transform the first member into the second:

11. $\dfrac{1+\sin\theta-\cos 2\theta}{\cos\theta+\sin 2\theta}=\tan\theta.$

12. $\dfrac{1+\cos\theta+\cos 2\theta}{\sin\theta+\sin 2\theta}=\cot\theta.$

13. $(\sqrt{1+\sin\alpha}+\sqrt{1-\sin\alpha})^2=4\cos^2\frac{1}{2}\alpha.$

14. $(\sqrt{1+\sin\alpha}-\sqrt{1-\sin\alpha})^2=4\sin^2\frac{1}{2}\alpha.$

15. $\tan\left(\dfrac{\pi}{4}+\alpha\right)-\tan\left(\dfrac{\pi}{4}-\alpha\right)=2\tan 2\alpha.$

16. $\cot\left(\dfrac{\pi}{4}+\alpha\right)-\cot\left(\dfrac{\pi}{4}-\alpha\right)=-2\tan 2\alpha.$

Find the values of α which satisfy the following equations:

17. $(2+\sqrt{3})(1-\sin 2\alpha)-2\cos^2 2\alpha=0.$

18. $\sin 2\alpha+2\cos 2\alpha=1.$

19. $4\sec^2 2\alpha+\tan 2\alpha=7.$

20. $\csc 2\alpha+\cot 2\alpha=2.$

Show that

21. $\tan^{-1}\dfrac{x-2}{3}=\cos^{-1}\dfrac{3}{\sqrt{x^2-4x+13}}.$

22. $\arctan\dfrac{2}{\sqrt{x^2+2x-3}}=\operatorname{arccsc}\dfrac{x+1}{2}.$

23. Find $\sin\left(\dfrac{\pi}{2}-2\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}\right).$

24. Find $\sin\left(\sin^{-1}m+\tan^{-1}\dfrac{\sqrt{1-m^2}}{m}\right).$

25. Find $\sin\left(\arccos(1-a)-2\arctan\sqrt{\dfrac{a}{2-a}}\right).$

26. Find $\cos(\arccos(1-2a)-2\arcsin\sqrt{a}).$

69. Conversion formulas for products. Adding the two first formulas of Arts. 63 and 65, we have

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta,$$

or, reversing and dividing by 2,

$$\boldsymbol{\sin\alpha\cos\beta=\tfrac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)].}\qquad(1)$$

If we subtract, instead of adding, we get

$$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta,$$

or

$$\boldsymbol{\cos\alpha\sin\beta=\tfrac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)].}\qquad(2)$$

Treating the two second formulas in like manner, we obtain

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)], \qquad (3)$$

and

$$\sin\alpha\sin\beta = -\tfrac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]. \qquad (4)$$

By means of these formulas, products of sines and cosines are expressed as sums or differences. By successive applications higher powers and products are reducible to expressions linear in sines and cosines. The same transformations may often be effected by application of the formulas of Art. 67, written in the form

$$\sin\alpha\cos\alpha = \tfrac{1}{2}\sin 2\alpha, \qquad (5)$$

$$\sin^2\alpha = \tfrac{1}{2}(1 - \cos 2\alpha), \qquad (6)$$

$$\cos^2\alpha = \tfrac{1}{2}(1 + \cos 2\alpha). \qquad (7)$$

EXERCISE XXII

Reduce the following products to linear expressions:

1. $\sin 5\alpha \cos 3\alpha$.

2. $\cos 6\alpha \sin 4\alpha$.

3. $\sin 7\alpha \sin 3\alpha$.

4. $\cos 2\alpha \cos 5\alpha$.

5. $\sin^2\alpha \cos\alpha$.

6. $\sin\alpha \cos^3\alpha$.

7. $\cos^4\alpha$.

8. $\sin^4\alpha$.

9. $\cos^3\alpha \sin^3\alpha$.

10. $\sin^2\alpha \cos^2\alpha$.

Show that

11. $\cos\alpha\sin(\beta-\gamma) + \cos\beta\sin(\gamma-\alpha) + \cos\gamma\sin(\alpha-\beta) = 0$.

12. $\sin(\beta-\gamma)\sin(\alpha-\delta) + \sin(\gamma-\alpha)\sin(\beta-\delta) + \sin(\alpha-\beta)\sin(\gamma-\delta) = 0$.

13. $\sin\frac{3\pi}{5}\cos\frac{\pi}{5} + \sin\frac{2\pi}{5}\cos\frac{4\pi}{5} = 0$.

14. $2\cos\frac{5\pi}{8}\cos\frac{\pi}{4} + \sin\frac{\pi}{8} + \cos\frac{5\pi}{8} = 0$.

Solve for α, making use of Art. 10.

15. $\cos(50^\circ + \alpha)\sin(50^\circ - \alpha) - \cos(40^\circ - \alpha)\sin(40^\circ + \alpha) = 0$.

16. $\sin(70^\circ + \alpha)\sin(70^\circ - \alpha) + \sin(20^\circ + \alpha)\sin(20^\circ - \alpha) = 0$.

Solve for α, making use of Art. 69.

17. $\cos 3\alpha + \cos 9\alpha = 0$.

18. $\sin 5\alpha - \sin 10\alpha = 0$.

19. $\cos(\alpha+\theta)\cos(\alpha-\theta)+\cos(3\alpha+\theta)\cos(3\alpha-\theta)=\cos 2\theta.$

20. $\sin(\alpha+\theta)\cos(\alpha-\theta)+\sin(3\alpha+\theta)\cos(3\alpha-\theta)=\sin 2\theta.$

70. Conversion formulas for sums and differences. In the process of deriving the formulas of the last article, before reversing and dividing by 2, substitute $\alpha+\beta=\phi$, $\alpha-\beta=\theta$, so that

$$\alpha=\frac{\phi+\theta}{2},\quad \beta=\frac{\phi-\theta}{2}.$$

We then obtain the following formulas:

$$\sin\phi+\sin\theta=2\sin\frac{\phi+\theta}{2}\cos\frac{\phi-\theta}{2}, \tag{1}$$

$$\sin\phi-\sin\theta=2\cos\frac{\phi+\theta}{2}\sin\frac{\phi-\theta}{2}, \tag{2}$$

$$\cos\phi+\cos\theta=2\cos\frac{\phi+\theta}{2}\cos\frac{\phi-\theta}{2}, \tag{3}$$

$$\cos\phi-\cos\theta=-2\sin\frac{\phi+\theta}{2}\sin\frac{\phi-\theta}{2}. \tag{4}$$

These formulas serve to effect transformations converse to those mentioned in Art. 69.

71. Multiple angles. In the formula for $\sin(\alpha+\beta)$ put $\beta=2\alpha$. Then

$$\begin{aligned}\sin 3\alpha&=\sin\alpha\cos 2\alpha+\cos\alpha\sin 2\alpha\\&=\sin\alpha-2\sin^3\alpha+2\sin\alpha\cos^2\alpha\\&=3\sin\alpha-4\sin^3\alpha.\end{aligned}$$

Again,

$$\begin{aligned}\cos 3\alpha&=\cos\alpha\cos 2\alpha-\sin\alpha\sin 2\alpha\\&=2\cos^3\alpha-\cos\alpha-2\sin^2\alpha\cos\alpha\\&=4\cos^3\alpha-3\cos\alpha.\end{aligned}$$

In like manner the other functions of 3α and, by repeating the process, the functions of any integral multiple of α may be expressed in terms of functions of α.

EXERCISE XXIII

Show that

1. $\dfrac{\sin 6\alpha+\sin 4\alpha}{\cos 6\alpha+\cos 4\alpha}=\tan 5\alpha.$

2. $\dfrac{\cos 3\alpha-\cos 5\alpha}{\sin 3\alpha+\sin 5\alpha}=\tan\alpha.$

3. $\frac{\sin 7\alpha - \sin 5\alpha}{\cos 7\alpha + \cos 5\alpha} = \tan \alpha.$

4. $\frac{\cos 4\alpha - \cos 2\alpha}{\sin 4\alpha - \sin 2\alpha} = -\tan 3\alpha.$

5. $\frac{\sin \alpha - \sin \beta}{\sin \alpha + \sin \beta} = \cot \frac{\alpha + \beta}{2} \tan \frac{\alpha - \beta}{2}.$

6. $\frac{\cos \alpha + \cos \beta}{\cos \alpha - \cos \beta} = -\cot \frac{\alpha + \beta}{2} \cot \frac{\alpha - \beta}{2}$

7. $\frac{\cos 3\theta + 2\cos 5\theta + \cos 7\theta}{\sin 3\theta + 2\sin 5\theta + \sin 7\theta} = \cot 5\theta.$

8. $\frac{\sin \theta - 2\sin 4\theta + \sin 7\theta}{\cos \theta - 2\cos 4\theta + \cos 7\theta} = \tan 4\theta.$

Solve the following equations:

9. $\cos\theta + \cos 5\theta = \cos 3\theta.$

10. $\sin\theta + \sin 5\theta = \sin 3\theta.$

11. $\sin 2\theta + 2\sin 4\theta + \sin 6\theta = 0.$

12. $\cos 3\theta + 2\cos 4\theta + \cos 5\theta = 0.$

Derive the formulas for:

13. $\cot 3\alpha$. (In terms of $\cot \alpha$.)

14. $\tan 3\alpha$. (In terms of $\tan \alpha$.)

15. $\sin 4\alpha$.

16. $\cos 4\alpha$.

Solve the equations:

17. $\sin 3\alpha = \sqrt{2}\sin 2\alpha.$

18. $\sqrt{3}\cos 3\alpha + 2\sin 2\alpha = 0.$

19. $\cos 3\alpha = \cos\alpha \cos 2\alpha.$

20. $\sin 3\alpha = \sin\alpha \cos 2\alpha.$

CHAPTER IX

ANALYTIC TRIGONOMETRY

The foregoing chapters constitute an introduction to the elementary principles of trigonometry. The student ought now to be prepared for a more advanced study of the theory of the trigonometric functions, which may be entitled analytic trigonometry. It is beyond the scope of this book to consider more than a few of the most important topics which might be discussed under this head. For a more extended treatment the student is referred to the treatises by Henrici and Treutlein, Hobson, Lock, Loney, Todhunter, and others, and, of course, to articles in the various mathematical journals.

72. Limits of θ/sin θ and θ/tan θ as θ approaches zero. Let θ be an acute angle measured in radians. Construct, as in Fig. 69, the angle $XOP = \theta$, repeated symmetrically as XOQ. Draw through P the arc PAQ with center O, the chord PMQ, and the broken or double tangent PTQ. Then

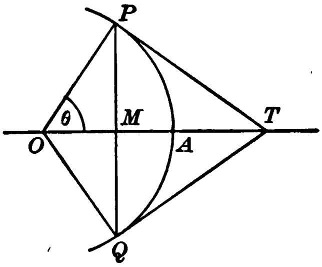

FIG. 69.

$$\frac{AP}{OP} = \theta, \qquad \frac{MP}{OP} = \sin\theta, \qquad \frac{TP}{OP} = \tan\theta.$$

By elementary geometry,

$$PMQ < PAQ < PTQ.$$

Whence, dividing by 2 and by OP,

$$\sin\theta < \theta < \tan\theta. \tag{1}$$

Dividing equation (1) through by $\sin\theta$, we have

$$1 < \frac{\theta}{\sin\theta} < \sec\theta.$$

Now in Art. 12 it was proved that as θ approaches the limit 0, $\cos\theta$ and its reciprocal $\sec\theta$ approach the limit 1. Thus, the value of $\theta/\sin\theta$ is always intermediate between 1 and a number that approaches the limit 1, as θ approaches 0. The ratio $\theta/\sin\theta$

must, therefore, approach the limit 1 at the same time. This is expressed symbolically by writing

$$\lim_{\theta \doteq 0} \left(\frac{\theta}{\sin \theta}\right) = 1.$$

Again, dividing equation (1) through by $\tan \theta$, we get

$$\cos \theta < \frac{\theta}{\tan \theta} < 1.$$

Now as θ approaches 0, $\cos \theta$ approaches 1, and hence, as before, $\theta / \tan \theta$ approaches the limit 1 at the same time. Symbolically,

$$\lim_{\theta \doteq 0} \left(\frac{\theta}{\tan \theta}\right) = 1.$$

NOTE. — Since $\sin \theta$ and $\tan \theta$ both approach 0 along with θ, it might seem that they therefore approach equality, and then the theorems would follow. The fallacy of assuming that the limiting form $\frac{0}{0}$ has the value 1 will appear on considering the following instances. The circumference and area of a circle approach zero simultaneously with the radius. We have, however, the general relations

$$\frac{\text{Circumference}}{\text{Radius}} = \frac{2\pi r}{r} = 2\pi = 6.28318 \cdots,$$

$$\frac{\text{Area}}{\text{Radius}} = \frac{\pi r^2}{r} = \pi r = 3.14159 \cdots r.$$

Now when r approaches the limit 0, the limit of the first ratio is the constant 2π, and the limit of the second ratio is 0.

The limiting form $\frac{0}{0}$ will be discussed at length in calculus. (See Townsend and Goodenough's "First Course in Calculus," Art. 13.)

EXAMPLE. If θ is increased by an angle δ, let it be required to determine the limit of the ratio of the consequent increase in $\sin \theta$ to the increment δ of θ, as that increment δ approaches zero. By Art. 70, we have

$$\frac{\sin(\theta + \delta) - \sin \theta}{\delta} = \frac{2 \cos\left(\theta + \frac{\delta}{2}\right) \sin \frac{\delta}{2}}{\delta},$$

$$= \cos\left(\theta + \frac{\delta}{2}\right) \cdot \frac{\sin \frac{\delta}{2}}{\frac{\delta}{2}}.$$

Now when δ approaches 0, $\cos\left(\theta+\frac{\delta}{2}\right)$ approaches $\cos\theta$ and

$$\frac{\sin\frac{\delta}{2}}{\frac{\delta}{2}} \text{ approaches } 1.$$

Hence

$$\lim_{\delta \doteq 0} \frac{\sin(\theta+\delta)-\sin\delta}{\delta} = \cos\theta.$$

It will be noticed that the numerator and denominator approach 0 simultaneously, but that the limit of the value of their ratio is a number somewhere between -1 and $+1$, and depending upon the value of θ.

EXAMPLES

In like manner find the limits as $\delta \doteq 0$, of

1. $\frac{\cos(\theta+\delta)-\cos\theta}{\delta}$.
2. $\frac{\sec(\theta+\delta)-\sec\theta}{\delta}$. (SUGGESTION. Express in terms of cosine.)
3. $\frac{\csc(\theta+\delta)-\csc\theta}{\delta}$.
4. $\frac{\tan(\theta+\delta)-\tan\theta}{\delta}$. (SUGGESTION. Express in terms of sine and cosine.)
5. $\frac{\cot(\theta+\delta)-\cot\theta}{\delta}$.

73. De Moivre's theorem. If we adopt the customary notation $i=\sqrt{-1}$, so that $i^2=-1$, we have, on performing the multiplication,

$$\begin{aligned}(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta \\ &\quad + i(\sin\alpha\cos\beta+\cos\alpha\sin\beta) \\ &= \cos(\alpha+\beta)+i\sin(\alpha+\beta), \qquad (1)\end{aligned}$$

a relation which holds for all values of α and β, whether positive or negative.

Putting $\beta=\alpha$, we get

$$(\cos\alpha+i\sin\alpha)^2=\cos 2\alpha+i\sin 2\alpha.$$

Again, putting $\beta=2\alpha$ in equation (1) and making use of the relation just established, we get

$$\begin{aligned}(\cos\alpha+i\sin\alpha)^3 &= (\cos\alpha+i\sin\alpha)(\cos 2\alpha+i\sin 2\alpha) \\ &= \cos 3\alpha+i\sin 3\alpha.\end{aligned}$$

Repetition of this process proves the relation

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha \qquad (2)$$

for all positive, integral values of n.

It is evident, upon multiplying, that

$$(\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha) = 1,$$

whence

$$(\cos\alpha + i\sin\alpha)^{-1} = \cos\alpha - i\sin\alpha.$$

Suppose n to be a negative integer. Let $n = -m$, where m is a positive integer. Now

$$\begin{aligned}(\cos\beta - i\sin\beta)^{-m} &= (\cos\beta + i\sin\beta)^m\\ &= \cos m\beta + i\sin m\beta.\end{aligned}$$

Substituting $m = -n$ and $\beta = -\alpha$, we get

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha,$$

true also for negative integral values of n.

Suppose n to be a fraction, either positive or negative. Let $n = \frac{r}{s}$, where r and s are integers. Now

$$(\cos\beta + i\sin\beta)^{\frac{r}{s}} = (\cos r\beta + i\sin r\beta)^{\frac{1}{s}}.$$

Raising both members to the sth power,

$$(\cos s\beta + i\sin s\beta)^{\frac{r}{s}} = \cos r\beta + i\sin r\beta.$$

Introducing $\frac{r}{s} = n$, and putting $s\beta = \alpha$, so that $r\beta = \frac{r}{s}\cdot s\beta = n\alpha$,

we get $\qquad (\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha.$

This relation, therefore, holds for all rational values of n. By an argument involving the method of limits it can be proved also for all irrational values of n. This is De Moivre's theorem, an instrument of great importance in some branches of mathematics.

EXAMPLE. An illustration of its use is afforded by applying it to the derivation of the formulas for the sines and cosines of multiple angles. Thus

$$\begin{aligned}\cos 3\alpha + i\sin 3\alpha &= (\cos\alpha + i\sin\alpha)^3\\ &= \cos^3\alpha + 3i\cos^2\alpha\sin\alpha - 3\cos\alpha\sin^2\alpha - i\sin^3\alpha.\end{aligned}$$

On equating the real terms on each side, and also the imaginary terms, separately, we have at once

$$\begin{aligned}\cos 3\alpha &= \cos^3\alpha - 3\cos\alpha\sin^2\alpha\\ &= 4\cos^3\alpha - 3\cos\alpha.\\ \sin 3\alpha &= 3\cos^2\alpha\sin\alpha - \sin^3\alpha\\ &= 3\sin\alpha - 4\sin^3\alpha.\end{aligned}$$

The functions of 4α and of higher multiples of α are as readily found. The simplicity and beauty of the method appears on comparison with that of Art. 71.

EXAMPLES

1. Show that $\dfrac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta} = \cos(\alpha-\beta) + i\sin(\alpha-\beta)$.

2. Show that $\left(\cos\dfrac{2\pi+\alpha}{n} + i\sin\dfrac{2\pi+\alpha}{n}\right)^n = \cos\alpha + i\sin\alpha$.

3. Show that $\left(\cos\dfrac{2k\pi+\alpha}{n} + i\sin\dfrac{2k\pi+\alpha}{n}\right)^n = \cos\alpha + i\sin\alpha$, where k is any integer.

4. Show that the angle $\dfrac{2k\pi+\alpha}{n}$ has n different values as k takes the successive values, $0, 1, 2, \cdots n-1$ (n being a positive integer). Show also that for all integral values of k outside these limits, the terminal sides of the angles coincide with those of the n angles already found.

5. Since $\cos 0 + i\sin 0 = 1$, find the n different nth roots of 1, of which all but one are imaginary. Making use of the tables of natural sines and cosines compute for $n = 2, 3, 4, 6$.

6. Since $\cos\pi + i\sin\pi = -1$, find the n different nth roots of -1, of which all but one are imaginary when n is odd, and all imaginary when n is even. Compute for $n = 2, 3, 4, 6$.

74. Graphical representation of complex numbers. An interesting application of De Moivre's theorem is found in the graphical representation of complex numbers, devised by Wessel, a Danish mathematician, and published by Argand in 1608. The treatment of this topic belongs rather to the courses in algebra and function theory. (See Rietz and Crathorne's "Algebra.") Only so much of the rudiments of the method will be developed here as possess a trigonometric interest.

A pure imaginary is an indicated square root of a negative number. A complex number is an indicated sum of a real number and a pure imaginary. All pure imaginaries can be expressed in the form yi, and all complex numbers in the form $x+yi$. Here $i=\sqrt{-1}$, so that $i^2=-1$; while x and y are real numbers, either rational or irrational.

Argand's method makes use of a pair of mutually perpendicular axes. The Argand diagram must not, however, be confused with the Cartesian scheme of coördinates.

All real numbers, rational or irrational, are represented by distances from the origin to points in the horizontal axis, called now the axes of reals, positive to the right, negative to the left. To every real number corresponds a point in this axis, and conversely, to every point in this axis corresponds a real number. Thus there is said to be a one-to-one correspondence between the totality of real numbers and the totality of points in the line.

All pure imaginaries are represented by distances from the origin to points in the vertical axis, now called the axis of imaginaries, points above and below the origin giving, respectively, positive and negative coefficients for the imaginary unit factor $i \equiv \sqrt{-1}$. Here again there exists a one-to-one correspondence between the totality of pure imaginaries and the totality of points in the vertical axis.

Notice that the origin alone, of all points in the plane, is on both axes. The number zero belongs to both systems. With this single exception, no pure imaginary can equal a real number, since the directions of the two axes are essentially different.

In order to represent the complex number $x+yi$ recourse must be had to the method of adding coplanar but non-collinear directed line segments employed in the graphical composition and resolution of forces in physics. Since directed line segments may undergo translation, the segment yi may be placed with its initial point upon the terminus of the segment x. The complex number is therefore represented by the right line segment (radius vector) v from the origin to the resulting terminus of the segment yi. For $y=0$ we have real numbers, for $x=0$ we have pure imaginaries.

As the lengths of the horizontal segment x and the vertical segment y measure respectively the magnitudes of the reals and the pure imaginaries, so the length of the radius vector v may be said to measure the absolute magnitude of the complex number $v=x+yi$. This is called the absolute or numerical value of v, and is denoted by the letter r. Evidently all points on the unit circle about the origin possess the absolute value 1.

The directed line segment, or radius vector, v makes in general an oblique angle with the axis of reals, and its direction is determined by the angle it forms with the positive axis of reals. This angle is denoted by θ, and is called the *amplitude* of the complex number. All points lying on the same radius have a common amplitude, while radii vectores extending from the origin in opposite directions have amplitudes differing by π. All positive real numbers have the amplitude 0; negative reals, π; pure imaginaries, $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

The right triangle formed by x, y, and v yields the relations

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\frac{v}{x},$$

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

We may write interchangeably,

$$v, \text{ or } x + yi, \text{ or } r(\cos\theta + i\sin\theta).$$

The expression $\cos\theta + i\sin\theta$ consequently denotes a unit segment (complex unit) with the amplitude θ, while r is a purely arithmetical factor.

Conjugate complex numbers, $x + yi$ and $x - yi$, evidently have the same absolute value and amplitudes which are negatives of each other.

Addition is effected graphically by placing the initial point of the second segment upon the terminus of the first and connecting the initial point of the first to the terminus of the second. Thus in Fig. 70,

$$\begin{aligned} v &= v_1 + v_2 \\ &= x_1 + iy_1 + x_2 + iy_2 \\ &= (x_1 + x_2) + i(y_1 + y_2). \end{aligned}$$

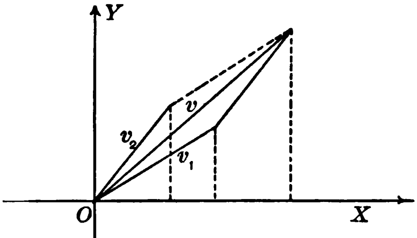

FIG. 70.

The values of r and θ in terms of r_1, r_2, θ_1 and θ_2 are readily determined, but exhibit little of present interest. Suffice it to point out that

$$r < r_1 + r_2,$$

$$\theta \neq \theta_1 + \theta_2.$$

Subtraction reduces at once to addition on reversing the subtrahend segment.

On attacking the problem of multiplication, we must define the product of a directed rectilinear segment by the imaginary unit i as a segment of equal length turned through a positive right angle. Thus $v = x + iy = r(\cos\theta + i\sin\theta)$ multiplied by i gives

$$v' = -y + ix = r\left[\cos\left(\frac{\pi}{2} + \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)\right].$$

The absolute value is unchanged, while the amplitude is increased by $\frac{\pi}{2}$. This is consistent with the original scheme of representation, since reals multiplied by i give pure imaginaries, and these multiplied by i give -1 times the original, *i.e.* the original radius vector reversed.

Multiplying a directed segment by a positive real number simply stretches it, multiplying its length and leaving its direction unchanged. Multiplying

$$v = x + iy = r(\cos\theta + i\sin\theta) \text{ by } k, \text{ we get}$$

$$v' = kv = kx + iky = kr(\cos\theta + i\sin\theta).$$

The absolute value is multiplied by the factor k, while the amplitude is unchanged.

Multiplication of one complex number by another is effected by combining the two processes just described, applying the associative and distributive laws. Thus

$$\begin{aligned} v = v_1 \cdot v_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\ &= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2) \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \end{aligned}$$

Using the other notation and applying De Moivre's theorem,

$$\begin{aligned} v = v_1 \cdot v_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2 \cdot [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \end{aligned}$$

Figure 71 illustrates the multiplication of $5 - 2i$ by $2 + 3i$. The product is shown to be $16 + 11i$. We have then the law that the absolute value of the product of two complex numbers equals the product of their absolute values, while the amplitude of the product equals the sum of their amplitudes.

The inverse process of division is readily performed, with the result

$$v=\frac{v_1}{v_2}=\frac{x_1+iy_1}{x_2+iy_2}=\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}+i\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2},$$

or

$$v=\frac{v_1}{v_2}=\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)},$$

$$v=\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)].$$

The absolute value of the quotient is equal to the quotient of the absolute values, while the amplitude of the quotient is equal to the difference of the amplitudes.

We have further,

$$v=v_1^n=r_1^n(\cos n\theta_1+i\sin n\theta_1).$$

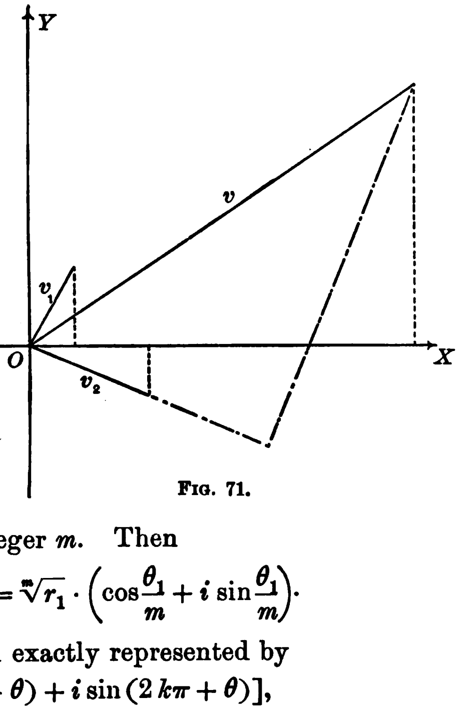

FIG. 71.

The absolute value of the power is equal to the power of the absolute value, while the amplitude of the power is equal to the amplitude of the number multiplied by the index of the power. Here "power" is used to denote the result of affecting the number by the exponent *n*, whatever the value of *n*. This includes both involution and evolution. In particular let *n* be the reciprocal of a positive integer *m*. Then

$$v=\sqrt[m]{v_1}=v_1^{\frac{1}{m}}=\sqrt[m]{r_1}\cdot\left(\cos\frac{\theta_1}{m}+i\sin\frac{\theta_1}{m}\right).$$

But v_1 is just as well and exactly represented by

$$r[\cos(2k\pi+\theta)+i\sin(2k\pi+\theta)],$$

where *k* is any integer. Thus the *m*th root just found is only one of an infinite number, all given by the form

$$v_1^{\frac{1}{m}}=\sqrt[m]{r_1}\left[\cos\frac{2k\pi+\theta_1}{m}+i\sin\frac{2k\pi+\theta_1}{m}\right],$$

in which k assumes all integral values. This form gives m different values for the root, corresponding to $k = 0, 1, 2, \cdots m - 1$. All the others are repetitions of these m roots, since the terminal sides of all the other amplitude angles will coincide with the terminal sides of the m amplitudes specified.

Hence every complex number has m different mth roots, whose common absolute value is the arithmetical mth root of the absolute value of the number, while their amplitudes have the m different values,

$$\frac{\theta_1}{m},\ \frac{2\pi + \theta_1}{m},\ \frac{4\pi + \theta_1}{m}, \ldots,\ \frac{2(m-1)\pi + \theta_1}{m},$$

all less than 2π.

In the special case of any positive real number x_1, whose amplitude is therefore zero, we obtain m different mth roots with the common absolute value $\sqrt[m]{r_1}$, which is called the principal value of $\sqrt[m]{x_1}$, and the m different amplitudes,

$$0,\ \frac{2\pi}{m},\ \frac{4\pi}{m},\ \frac{6\pi}{m}, \ldots \frac{2(m-1)\pi}{m}.$$

Only one of these is real, the first, and it is called the principal mth root of the positive real number.

The student should construct figures to illustrate the foregoing theorems. Still another analytic notation for complex numbers will be brought out in Art. 75.

EXAMPLES

1. Represent by Argand's diagrams the numbers 2, -3, $3i$, $-4i$, $3+5i$, $4-3i$, $-2+i$, $-5-3i$, $4+\sqrt{-3}$, $\sqrt{5}-\sqrt{-2}$.

2. Write the numbers the termini of whose radii vectores have the Cartesian coördinates $(3, 4)$, $(-3, 2)$, $(7, -3)$, $(-5, -2)$, $(6, 0)$, $(0, 5)$, $(-2, 0)$, $(0, -6)$, $(0, 0)$, $(\sqrt{3}, \sqrt{5})$.

3. Find the absolute values and the amplitudes (expressed in degrees and minutes) of the numbers in examples 1 and 2.

4. Describe the situation of the number points which have: (1) the common absolute value 3; (2) the common amplitude 30°; (3) the amplitudes 45° and 225°.

5. Perform graphically: $(3+4i) + (7-2i)$; $(-3+2i) + (6-3i)$; $(7-3i) - (4+2i)$; $(3-2i) - (-6-3i)$; $(5+2i) + (3-4i) - (6-3i)$.

6. Perform graphically, taking the first factor in each case as the multiplier: $3 \cdot (5+2i)$; $i \cdot (3+5i)$; $2i \cdot (6-3i)$; $-4 \cdot (2+5i)$; $-6i \cdot (3+2i)$; $(4+2i) \cdot (3+4i)$; $(3+4i) \cdot (4+2i)$.

7. Construct the quotient of $\frac{21+i}{3+2i}$; $\frac{6-17i}{4-3i}$.

8. Construct: $(3+2i)^2$; $\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3$; $(1-i\sqrt{3})^3$; i^4.

9. Find by construction: $\sqrt{7-24i}$; $\sqrt[4]{-119+120i}$; $(-5+12i)^{\frac{1}{2}}$; $\sqrt{-4}$; $\sqrt[3]{8}$; $\sqrt[4]{16}$.

10. Write the general solution of the binomial equation: $x^n - a^n = 0$.

11. Find *all* the roots of the equations $x^2-1=0$; $x^2+1=0$; $x^3-1=0$; $x^3-8=0$.

75. Exponential values of the trigonometric functions. The first form of De Moivre's theorem, Art. 73, Eq. (1), may be written symbolically,

$$F(\alpha)\cdot F(\beta) = F(\alpha+\beta),$$

which is read, function of α times (the same) function of β equals (the same) function of $(\alpha+\beta)$; or, the product of the (same) functions of two numbers equals the (same) function of the sum of the two numbers. Now this is identically the characteristic relation or law governing the exponential function, that is, a function of the form a^x; thus,

$$a^\alpha \cdot a^\beta = a^{\alpha+\beta}.$$

For reasons discussed in Art. 77, it is found that instead of the more general function a^α, we must place

$$\cos\alpha + i\sin\alpha = e^{i\alpha}, \tag{1}$$

where $e = 2.71828183 \cdots$ is the base of the Naperian, or natural, system of logarithms given in Art. 23.

Note that the law of exponents, derived for positive integral exponents, and assumed to hold also for negative, fractional, and irrational exponents, is still further assumed for exponents which are pure imaginaries and complex numbers. As in the former cases, the significance must be determined in conformity to the action of the assumed law. Indeed, the *law* defines the *function*.

Since $\cos\alpha - i\sin\alpha = \frac{1}{\cos\alpha + i\sin\alpha}$, we have also

$$\cos\alpha - i\sin\alpha = e^{-i\alpha}. \tag{2}$$

Adding and dividing by 2, we obtain

$$\cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2}; \tag{3}$$

again, subtracting and dividing by $2i$,

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}. \tag{4}$$

These values were first given by Euler in 1743. Starting from these two exponential values as fundamental definitions, and defining further

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha},\quad \cot\alpha = \frac{1}{\tan\alpha},\quad \sec\alpha = \frac{1}{\cos\alpha},\quad \csc\alpha = \frac{1}{\sin\alpha},$$

it is possible to develop all the laws and formulas of trigonometry as contained in Arts. 59 and 63–71, quite apart from any geometric meaning attached to the functions or their argument α. The analogous derivation of those trignometric theorems dependent on the periodicity of the trigonometric functions involves the periodicity of the logarithm, and is therefore postponed until the later mathematical study of the student.

A third notation for complex numbers now becomes manifest; for

$$v = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

The consequent theorems regarding the absolute values and amplitudes of products, quotients, powers, and roots follow readily, and should be worked out by the student.

Examples

1. Find the exponential values of $\tan\alpha$, $\cot\alpha$, $\sec\alpha$, $\csc\alpha$.

2. Derive from the exponential values the laws $\sin^2\alpha + \cos^2\alpha = 1$, etc., of Art. 59.

3. Derive from the exponential values the formulas of Arts. 63–71.

4. Derive from the exponential notation the laws for the absolute values and amplitudes of products, quotients, powers, and roots of complex numbers.

76. Hyperbolic functions. Closely allied to Euler's forms of the last article are the two interesting and important forms,

$$\frac{e^{\alpha} + e^{-\alpha}}{2} \text{ and } \frac{e^{\alpha} - e^{-\alpha}}{2}.$$

They are called, by analogy, the hyperbolic cosine and hyperbolic sine. Thus, employing the customary notation,

$$\cosh\alpha = \frac{e^{\alpha} + e^{-\alpha}}{2},\quad \sinh\alpha = \frac{e^{\alpha} - e^{-\alpha}}{2}.$$

The remaining hyperbolic functions are defined from these, just as in Art. 75 :

$$\tanh \alpha = \frac{\sinh \alpha}{\cosh \alpha},\ \coth \alpha = \frac{1}{\tanh \alpha},\ \operatorname{sech} \alpha = \frac{1}{\cosh \alpha},\ \operatorname{csch} \alpha = \frac{1}{\sinh \alpha}.$$

A very simple relation exists between the hyperbolic and the circular (*i.e.* ordinary trigonometric) functions. Evidently

$$\cosh \alpha = \cos i\alpha,$$
$$\sinh \alpha = -i \sin i\alpha,$$
$$\tanh \alpha = -i \tan i\alpha;$$

and conversely,

$$\cos \alpha = \cosh i\alpha,$$
$$\sin \alpha = -i \sinh i\alpha,$$
$$\tan \alpha = -i \tanh i\alpha.$$

To each formula of Chapter VIII corresponds a formula for the hyperbolic functions, which may be deduced either directly from the exponential definitions, or by substituting the values just given in the formulas for the circular functions. The student should derive these formulas by both methods.

The analogue to De Moivre's theorem is

$$(\cosh \alpha + \sinh \alpha)^n = \cosh n\alpha + \sinh n\alpha.$$

Cosh α and sinh α possess an imaginary period $2\pi i$, since $e^u = e^{u+2k\pi i}$, k being any integer. (See treatises on the theory of functions.)

77. Exponential and trigonometric series. In the present article values in the form of infinite series will be derived for certain exponential, logarithmic, and trigonometric functions. In the proof, however, the use of the binomial formula and the manipulation of the series introduce a lack of rigor requiring extended consideration in the subsequent courses in algebra, the calculus, and the theory of functions.

(1) *Exponential series.*

Expanding by the binomial formula,*

$$\left(1+\frac{x}{n}\right)^n = 1 + \frac{n}{1!}\cdot\frac{x}{n} + \frac{n(n-1)}{2!}\cdot\frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{x^3}{n^3} + \cdots$$

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* The symbol $\underline{|k}$, or $k!$, is used to denote the product $1\cdot 2\cdot 3 \cdots k$, where k is any positive integer, and is read "factorial k".

$$= 1 + 1 \cdot \frac{x}{1!} + 1 \cdot \left(1 - \frac{1}{n}\right) \cdot \frac{x^2}{2!} + 1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdot \frac{x^3}{3!} + \cdots.$$

Now as n becomes infinite, the binomial factors $\left(1 - \frac{1}{n}\right)$, $\left(1 - \frac{2}{n}\right)$, etc., all approach the common limit 1, and we shall have, in the limit,

$$e^x = \lim_{n \to \infty}\left[\left(1 + \frac{x}{n}\right)^n\right] = 1 + \frac{x}{1!} + \frac{x}{2!} + \frac{x}{3!} + \cdots. \qquad (1)$$

This series is convergent for all finite values of x. (See Rietz and Crathorne's "Algebra.")

For $x = 1$ we get

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \qquad (2)$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots.$$

The terms diminish rapidly in value and, when expressed decimally, the value of e is found to be $2.71828183 \cdots$.

The series for e^x is valid also for negative and imaginary values of x; thus, substituting successively $-x$, ix, and $-ix$ for x, we have

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots,$$

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + i\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right],$$

$$e^{-ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots - i\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right].$$

(2) *Logarithmic series.*

From the expansion just obtained for e^x can be derived a series for $\log_e(1 + y)$.

Since* $u^x = e^{x \log_e u}$,

we have $u^x = 1 + \frac{x}{1!}(\log_e u) + \frac{x^2}{2!}(\log_e u)^2 + \frac{x^3}{3!}(\log_e u)^3 + \cdots.$

Placing $u = 1 + y$,

$$(1 + y)^x = 1 + \frac{x}{1!}\log_e(1 + y) + \frac{x^2}{2!}[\log_e(1+y)]^2 + \frac{x^3}{3!}[\log_e(1 + y)]^3 + \cdots.$$

---

* Let $w = u^x$. Taking logarithms to base e, we have $\log_e w = x \log_e u$. Now taking exponentials to base e, $w = u^x = e^{x \log_e u}$.

Expanding the first member by the binomial formula,

$$(1+y)^x = 1 + \frac{x}{1!}y + \frac{x(x-1)}{2!}y^2 + \frac{x(x-1)(x-2)}{3!}y^3 + \cdots.$$

Picking out and equating the coefficients of x in the two expressions, the required expansion is obtained,

$$\log_e(1+y) = \frac{y}{1} - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots. \qquad (3)$$

This series is convergent for $-1 < y \leqq +1$.

If $w = \log_a u$, we have $a^w = u$; whence, taking logarithms to base e, $w \log_e a = \log_e u$. Therefore $w = \log_a u = \frac{1}{\log_e a} \cdot \log_e u$.

Substituting from (3) we see that

$$\log_a(1+y) = \frac{1}{\log_e a}\left(\frac{y}{1} - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\right). \qquad (4)$$

(3) *Trigonometric series.*

From De Moivre's theorem

$$\cos m\theta + i \sin m\theta = (\cos\theta + i\sin\theta)^m.$$

Expanding by the binomial formula and separating the real terms from the imaginary,

$$\cos m\theta + i\sin m\theta = \cos^m\theta - \frac{m(m-1)}{2!}\cos^{m-2}\theta\sin^2\theta$$

$$+ \frac{m(m-1)(m-2)(m-3)}{4!}\cos^{m-4}\theta\sin^4\theta \cdots$$

$$+ i\left(\frac{m}{1!}\cos^{m-1}\theta\sin\theta - \frac{m(m-1)(m-2)}{3!}\cos^{m-3}\theta\sin^3\theta + \cdots\right).$$

Equating separately the real and the imaginary parts,

$$\cos m\theta = \cos^m\theta - \frac{m(m-1)}{2!}\cos^{m-2}\theta\sin^2\theta + \cdots,$$

$$\sin m\theta = \frac{m}{1!}\cos^{m-1}\theta\sin\theta - \frac{m(m-1)(m-2)}{3!}\cos^{m-3}\theta\sin^3\theta + \cdots.$$

Place now

$$m\theta = \alpha, \text{ so that } \theta = \frac{\alpha}{m};$$

$$\cos\alpha = \cos^m\left(\frac{\alpha}{m}\right) - \frac{m(m-1)}{2!}\cos^{m-2}\left(\frac{\alpha}{m}\right)\sin^2\left(\frac{\alpha}{m}\right) + \cdots,$$

$$\sin\alpha = \frac{m}{1!}\cos^{m-1}\left(\frac{\alpha}{m}\right)\sin\left(\frac{\alpha}{m}\right)$$

$$- \frac{m(m-1)(m-2)}{3!}\cos^{m-3}\left(\frac{\alpha}{m}\right)\sin^3\left(\frac{\alpha}{m}\right) + \cdots.$$

Now let θ approach the limit zero and m become infinite, while still obeying the condition that $m\theta = \alpha$, where α remains finite. By Art. 12, $\cos\frac{\alpha}{m}$, $\cos^2\frac{\alpha}{m}$, etc., approach the limit 1 as m becomes infinite, and in the calculus the same is shown for $\cos^m\left(-\right)$, $\cos^{m-1}\left(\frac{\alpha}{m}\right)$, etc.

Again, $$m\sin\left(\frac{\alpha}{m}\right) = \cdot\frac{\sin\theta}{\theta},$$

$$m(m-1)\sin^2\left(\frac{\alpha}{m}\right) = \alpha(\alpha-\theta)\cdot\left(\frac{\sin\theta}{\theta}\right)^2,$$

$$m(m-1)(m-2)\sin^3\left(\frac{\alpha}{m}\right) = \alpha(\alpha-\theta)(\alpha-2\theta)\cdot\left(\frac{\sin\theta}{\theta}\right)^3, \text{ etc.}$$

Since $\lim\limits_{\theta \doteq 0}\frac{\sin\theta}{\theta} = 1$, the limits approached by these expressions, as $\theta \doteq 0$, are α, α^2, α^3, etc.

Making these substitutions, we obtain, in the limit,

$$\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots, \tag{5}$$

$$\sin\alpha = \frac{\alpha}{1!} - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \cdots. \tag{6}$$

These series are convergent for all values of α.

Tan α may also be expanded into the series

$$\tan\alpha = \frac{\alpha}{1} + \frac{\alpha^3}{3} + \frac{\alpha^5}{15} + \frac{17\,\alpha^7}{315} + \cdots. \tag{7}$$

It will be noticed that the series for $\cos\alpha$ contains only even powers of α, while those for $\sin\alpha$ and $\tan\alpha$ contain only the odd powers of α. (See the third example worked out in Art. 62.)

The assumption of Art. 75 may now be justified. For, on substituting for e^{ix}, e^{-ix}, $\sin x$, and $\cos x$ their expansions in series, we obtain

$$\cos x + i\sin x = e^{ix},$$

$$\cos x - i\sin x = e^{-ix},$$

and $$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

(4) *Hyperbolic series.*

From the analogous relations the expansions for the hyperbolic functions are readily obtained.

Thus
$$\cosh \alpha = \frac{e^{\alpha} + e^{-\alpha}}{2} = 1 + \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \frac{\alpha^6}{6!} + \cdots, \tag{8}$$

$$\sinh \alpha = \frac{e^{\alpha} - e^{-\alpha}}{2} = \frac{\alpha}{1!} + \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \frac{\alpha^7}{7!} + \cdots. \tag{9}$$

EXAMPLES

1. By substituting $-\alpha$ for α, find the series for $\sin(-\alpha)$, $\cos(-\alpha)$, $\tan(-\alpha)$, and by comparison, verify the corresponding relations of Art. 62.

2. By substituting $i\alpha$ for α, verify the relations of Art. 76, $\cosh \alpha = \cos i\alpha$, etc.

3. Using two terms of the expansions for $\sin \alpha$ and $\cos \alpha$, and retaining only powers of α below the fifth, obtain an approximate verification of the following formulas:
$\sin^2 \alpha + \cos^2 \alpha = 1$, $\sin(\alpha + \beta) = \cdots$, $\cos(\alpha + \beta) = \cdots$, $\sin 2\alpha = \cdots$, $\cos 2\alpha = \cdots$.

4. Do as required in example 3 for the hyperbolic functions.

5. Repeat examples 3 and 4, using three terms of the series and retaining powers of α below the seventh, thus arriving at a closer approximation.

78. Computation of trigonometric tables. The numerical values of the sine, cosine, and other trigonometric functions of angles from 0° to 90°, as tabulated in Table III, may be calculated by means of various trigonometric formulas, or better, by the use of the series derived in Art. 77.

Euler gave the following series, carrying the computation to 28 decimal places and a corresponding number of terms: Place $\alpha = m \cdot \frac{\pi}{2}$, in the series for $\sin \alpha$ and $\cos \alpha$; whence

$$\begin{aligned}\sin\left(m \cdot \frac{\pi}{2}\right) = {} & 1.570\,796\,m - 0.645\,964\,m^3 \\ & + 0.079\,693\,m^5 - 0.004\,682\,m^7 \\ & + 0.000\,160\,m^9 - 0.000\,004\,m^{11} \\ & + \cdots\cdots\cdots\cdots, \\ \cos\left(m \cdot \frac{\pi}{2}\right) = {} & 1.000\,000 \quad - 1.233\,700\,m^2 \\ & + 0.253\,699\,m^4 - 0.020\,863\,m^6 \\ & + 0.000\,919\,m^8 - 0.000\,025\,m^{10} \\ & + \cdots\cdots\cdots\cdots.\end{aligned}$$

We need to calculate the sines and cosines of angles up to 45° only, so that m is a fraction and always less than $\frac{1}{2}$. The terms of the series converge rapidly and a few terms suffice to give values correct to a small number of decimal places.

More extended discussion of this topic may be found in Hobson's "Trigonometry," Chap. IX; Todhunter's "Plane Trigonometry," Chap. X; and other advanced treatises on trigonometry.

The series for $\log(1+y)$ converges too slowly for convenient calculation, but a modified form is easily obtained. Manifestly

$$\log(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \cdots$$

and

$$\begin{aligned}\log\frac{1+y}{1-y} &= \log(1+y) - \log(1-y)\\ &= 2\left[\frac{y}{1} + \frac{y^3}{3} + \frac{y^5}{5} + \cdots\right].\end{aligned}$$

Place $\qquad y = \dfrac{1}{2v+1}$, whence $\dfrac{1+y}{1-y} = \dfrac{v+1}{v}$; then

$$\log\frac{v+1}{v} = 2\left(\frac{1}{2v+1} + \frac{1}{3(2v+1)^3} + \frac{1}{5(2v+1)^5} + \cdots\right),$$

or

$$\log(v+1) = \log v + 2\left[\frac{1}{2v+1} + \frac{1}{3(2v+1)^3} + \frac{1}{5(2v+1)^5} + \cdots\right].$$

This series converges rapidly and by it $\log 2$ can be computed from $\log 1 = 0$, $\log 3$ from $\log 2$, etc. Logarithms of composite numbers can be checked by adding the logarithms of their factors.

79. Proportional parts. In using the logarithmic and trigonometric tables it was assumed, as stated in Art. 26, that for small differences in the number, the differences in the logarithm are proportional to the differences in the number, and that likewise, for small differences in the angle, the differences in the sine (or other trigonometric function, or logarithmic function) are proportional to the differences in the angle.

We have

$$\begin{aligned}\log(x+\delta) - \log x &= \log\frac{x+\delta}{x} = \log\left(1+\frac{\delta}{x}\right)\\ &= \frac{\delta}{x} - \frac{\delta^2}{2x^2} + \frac{\delta^3}{3x^3} - \frac{\delta^4}{4x^4} + \cdots\\ &\doteq \frac{\delta}{x}. \qquad \text{(Approximately for small values of } \delta.)\end{aligned}$$

Therefore, we have approximately

$$\frac{\log(x+\delta_1)-\log x}{\log(x+\delta_2)-\log x}=\frac{\delta_1/x}{\delta_2/x}=\frac{\delta_1}{\delta_2}, \tag{1}$$

for small differences.

Again,

$$\sin(\delta+\theta)-\sin\theta=2\cos\left(\theta+\frac{\delta}{2}\right)\sin\frac{\delta}{2}$$

$$=\cos\theta\cdot\delta. \quad \text{(Approximately for small values of } \delta.)$$

Hence, approximately

$$\frac{\sin(\theta+\delta_1)-\sin\theta}{\sin(\theta+\delta_2)-\sin\theta}=\frac{\delta_1\cos\theta}{\delta_2\cos\theta}=\frac{\delta_1}{\delta_2}. \tag{2}$$

For the other functions, the proof follows exactly similar lines, and can easily be supplied by the student.

Full discussion along this line may be found in Loney's "Plane Trigonometry," Chap. XXX; Lock's "Higher Trigonometry," Chap. VIII; Hobson's "Trigonometry," Chap. IX; etc.

80. General inverse functions. In Art. 14 only acute angles were under consideration, so that the relations

$$m=\sin\alpha, \qquad \alpha=\arcsin m,$$

expressed a one-to-one correspondence. In other words, under the condition that

$$0^\circ \leqq \alpha \leqq 90^\circ, \qquad 0 \leqq m \leqq 1,$$

to each value of α there corresponds one and only one value of m and conversely.

On considering the general angle, it became evident that to any one angle there corresponds one and only one value of the sine (or other function), but that to one value of the sine (or other function) correspond many angles. We *define* arcsin m, arccos m, arctan m, etc., as the numerically smallest angle having the given sine, cosine, tangent, etc. It follows that arcsin m, arctan m, arccot m, arccsc m always lie between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, while arccos m and arcsec m always lie between 0 and $+\pi$. These are called the *principal* values of the general inverse functions Arcsin m, Arccos m, etc. As results of Arts. 61, 62, we may write, if k is any integer,

$$\begin{aligned}
\text{Arcsin } m &= 2\,k\pi + \arcsin m,\\
\text{Arccos } m &= 2\,k\pi + \arccos m,\\
\text{Arctan } m &= k\pi + \arctan m,\\
\text{Arccot } m &= k\pi + \text{arccot } m,\\
\text{Arcsec } m &= 2\,k\pi + \text{arcsec } m,\\
\text{Arccsc } m &= 2\,k\pi + \text{arccsc } m.
\end{aligned}$$

Similar relations exist for the inverse hyperbolic functions, the periods being imaginary, $2\,k\pi i$ and $k\pi i$.

From the relations of Art. 76 may be derived the following:

$$\begin{aligned}
\arccos m &= (\pm)\, i \text{ inv } \cosh m,\\
\arcsin m &= -\, i \text{ inv } \sinh im,\\
\arctan m &= -\, i \text{ inv } \tanh im;
\end{aligned}$$

and

$$\begin{aligned}
\text{inv } \cosh m &= (\pm)\, i \arccos m,\\
\text{inv } \sinh m &= -\, i \arcsin im,\\
\text{inv } \tanh m &= -\, i \arctan im.
\end{aligned}$$

81. Logarithmic values of inverse functions. Since the circular and hyperbolic functions are expressible as exponential functions, it would seem that the inverse functions should be expressible as logarithmic functions. Such is, indeed, the case, and the desired values may be found by solving for α the forms given in Arts. 75 and 76.

(1) *Circular functions.*

If, for example,

$$z = \sin \alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2\,i},$$

we have the quadratic equation in $e^{i\alpha}$,

$$e^{2i\alpha} - 2\,ize^{i\alpha} - 1 = 0,$$

whose roots are

$$e^{i\alpha} = iz \pm \sqrt{1 - z^2}.$$

Choosing the upper sign, and taking logarithms to base e, we get

$$i\alpha = \log\,(iz + \sqrt{1 - z^2}),$$

whence

$$\alpha = \arcsin z = -\,i \log\,(iz + \sqrt{1 - z^2})$$

gives the principal value of the arcsine.

(2) *Hyperbolic functions.*

As may be expected, the values of the inverse hyperbolic functions are real in form. Thus from

$$z = \sinh \alpha = \frac{e^{\alpha} - e^{-\alpha}}{2},$$

we get, on solving,

$$\alpha = \text{inv} \sinh z = \log (z + \sqrt{z^2 + 1}).$$

EXAMPLES

Obtain

1. $\text{arccos}\, z = -i \log (z + \sqrt{z^2 - 1})$.

2. $\text{arc} \tan z = \frac{1}{2} i \log \frac{1 + iz}{1 - iz}$.

3. $\text{inv} \cosh z = \log z (z + \sqrt{z^2 - 1})$.

4. $\text{inv} \tanh z = \frac{1}{2} \log \frac{1 + z}{1 - z}$.

REVIEW EXERCISES

1. To what quadrant do the following angles belong: 560°, 653°, 1030°, $-425°$, $-1260°$?

2. To what quadrant do the following angles belong: $\frac{2\pi^R}{5}$, $\frac{5\pi^R}{3}$, 8^R, $\frac{13\pi_R}{3}$, 25^R, $-\frac{7\pi^R}{4}$, $-\frac{10\pi^R}{3}$?

3. Reduce to radians: 75°, $-300°$, $-250°$, 2000°, 465° 20′.

4. Reduce to the degree system: 4^R, -6^R, $\frac{4\pi^R}{3}$, $\frac{8\pi^R}{5}$, $-\frac{7\pi^R}{2}$.

5. Find the lengths of the arcs subtended by the following angles at the center of a circle of radius 6: 45°, 120°, 270°, $\frac{3\pi^R}{4}$, $\frac{5\pi^R}{8}$, $\frac{5\pi^R}{3}$.

6. A polygon of n sides is inscribed in a circle of radius r. Find the length of the arc subtended by one side. Compute the numerical values if $r = 10$ and $n = 3, 4, 5, 6, 8$.

7. Taking the radius of the earth to be 4000 miles, find the difference in latitude of two points on the same meridian 300 miles apart.

8. Find the difference in longitude of two points on the equator 1200 miles apart.

9. Find the distance in degrees between two points, one of which is 800 miles due north of the other.

10. A city is surrounded by a circular belt line 5 miles in radius. How long will a train require to go at a speed of 20 miles an hour from a station due east of the center to one due northwest, if the motion is clockwise; if counterclockwise?

11. Find with the protractor the angles formed successively by the radii vectores of the points (3, 0), (2, 4), (-3, 5), (0, 6), (-4, 2), (-2, 1), (5, -3), (8, 0).

12. Find with the protractor the angles of the triangles formed by the abscissa, ordinate, and radius vector of each of the following points: (4, 4), (1, 3), (3, -3), (-2, 2), (-4, -8).

13. Find by measurement the coördinates of the point whose radius vector is 4 and makes an angle of 30° with the positive x-axis; 5 and 120°; 8 and 225°.

14. Find by measurement the length and inclination angle of the radii vectores of the points whose coördinates are (2, 5), (-5, 12), (-8, -15).

15. If the earth were assumed to be a plane, and one degree of latitude or longitude were 60 miles, what would be the distance and direction from a point in 20° N. lat., 60° E. long., to one in 50° N. lat., 30° E. long.; from a point in 30° S. lat., 15° E. long., to one in 45° N. lat., 40° W. long.?

NOTE. This assumption is made by navigators as a basis for what is known as Plane Sailing. In Great Circle Sailing the earth is considered a sphere. Let the student devise a system of coördinates for the latter.

16. Find by measurement the six trigonometric functions of 36°, 155°, 285°, – 130°.

17. Find by measurement the following angles: arccot $\frac{2}{3}$ and of 1st quadrant; arcsin $\frac{5}{8}$ and of 2d quadrant; arccos (-0.3) and of 3d quadrant; arcsec 0.6 and of 4th quadrant.

18. Find the lacking functions in the following table:

Angle	Sine	Cosine	Tangent	Cotangent	Secant	Cosecant	Quadrant
α	$-\frac{8}{17}$						III
β		$\frac{2}{3}$					IV
γ			$-\frac{1}{3}$				II
δ				$\frac{24}{7}$			III
θ					-2		III
ϕ						3	I

19. Find the value of $\frac{\cos\alpha}{1-\tan\alpha}+\frac{\sin\alpha}{1-\cot\alpha}$ if $\alpha=\arcsin\frac{3}{5}$ and of 2d quadrant.

20. Find the value of $\frac{\tan\beta+\sec\beta-1}{\tan\beta-\sec\beta+1}$ if $\beta=\text{arccsc}\left(-\frac{13}{5}\right)$ and of 3d quadrant.

21. Find the value of $\frac{\tan^2\alpha-\sin^2\alpha}{\cos^2\alpha}$ if $\alpha=\text{arccot}\left(-\frac{1}{2}\right)$ and of 4th quadrant.

22. Find the value of $\frac{1+\cos\gamma-2\sec\gamma}{3+\cos\gamma+2\sec\gamma}$ if $\gamma=\arctan\frac{9}{40}$ and of 1st quadrant.

23. Express $\frac{\sin^2\alpha+\cos^2\alpha}{\sec^2\alpha+\csc^2\alpha}$ in terms of $\tan\alpha$.

24. Express $\frac{4-\sin\beta-3\csc\beta}{\sin\beta-1-6\csc\beta}$ in terms of $\cos\beta$.

25. Express $\frac{(1-\cos\alpha)(1+\sec\alpha)}{(1-\sin\alpha)(1+\csc\alpha)}$ in terms of $\tan\alpha$.

26. In the following identity transform the first member into the second,

$$\frac{(1+\tan\gamma)(\cos\gamma-\cot\gamma)}{(1+\cot\gamma)(\sin\gamma-\tan\gamma)}=-\cot\gamma.$$

27. Show that $\frac{(1-\tan\alpha)(1+\cot\alpha)}{(1+\tan\alpha)(1-\cot\alpha)}=-1.$

28. Show that $\frac{\sin^3\beta+\cos^3\beta}{\sin^2\beta+\cos^2\beta}+\frac{\sin^3\beta-\cos^3\beta}{\sin^2\beta-\cos^2\beta}=-2\sin^2\beta\cos^2\beta.$

Solve the following equations and find the angle in degrees :

29. $4\sin^2\gamma-\tan^2\gamma=0.$

30. $2\tan^2\alpha-\sec\alpha=4.$

31. $4\csc\beta+\cot^2\beta=5.$

32. $\tan^2\gamma+3\cot^2\gamma=4.$

33. $\sin^2\alpha+\sin^2\beta=\frac{5}{4}$, $\cos^2\alpha+\cos^2\beta=0.$

34. $2\cos^2\alpha+\sin^2\beta=2$, $\sin\alpha+\cos^2\beta=0.$

35. For what range of values of α between 0 and 2π is $\sin\alpha+\cos\alpha$ positive; negative?

36. For what range of values of β between 0 and 2π is $\tan\beta-\cot\beta$ positive; negative?

37. Show that $\tan\gamma+\cot\gamma$ must always be numerically greater than unity.

38. Trace the variation of $\sin^2\theta$ as θ varies from 0 to 2π.

39. Trace the variation of $\cos^2\theta$ as θ varies from 0 to 2π.

40. Trace the variation of $1-\sin\theta$ as θ varies from 0 to 2π.

41. Trace the variation of $1-\cos\theta$ as θ varies from 0 to 2π.

42. Find by inspection $\log_8 .625$, $\log_{81} 27$, $\log_{25} .008$.

43. What numbers correspond to the following logarithms to base 9: -3, -2, -1.5, -1, 0, .5, 1, 2, 3?

44. In the formula $W=\frac{144n}{n-1}p_1v_1\left[\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}-1\right]$, which gives the work of an air compressor, find W when $n=1.3$, $p_1=14.7$, $p_2=72$, $v_1=6$.

45. Work the following with the slide rule:

(*a*) $\frac{.72\times137\times14}{372\times778}=?$ (*b*) $\left(\frac{120}{42}\right)^{1.3}=?$ (*c*) $\frac{42\sin27^\circ}{13.24}=?$

46. Solve for x: $5^{2x}=6$; $8^{x-1}=7$.

QUERY. Does the result depend on the base of the system of logarithms used?

47. Solve for x: $3^{2x}-4\cdot3^x+3=0$.

48. Find the amount of $2000 in 5 years at 4% compound interest.

49. At what rate, compound interest, will $45,000 amount in 8 years to $60,000?

50. In how many years will a city become three times its original size if it increases $\frac{1}{8}$ each year?

51. Derive the formulas of Art. 42 from those of Art. 40. [SUGGESTION. Multiply respectively by a, b, and $-c$ and add.]

52. Derive the formulas of Art. 40 from those of Art. 42. [SUGGESTION. Solve for $a \cos \beta$ and $b \cos \alpha$ and add.]

53. From the law of sines, Art. 41, show that $\frac{b-c}{b+c} = \frac{\sin \beta - \sin \gamma}{\sin \beta + \sin \gamma}$.

54. By applying the formulas of Art. 70 to the result obtained in example 53, derive the law of tangents of Art. 43.

55. From the formulas of Arts. 42 and 68 derive the results

$$\sin \frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}; \ \cos \frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}; \ \tan \frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

56. Draw the graph of $\sin \theta + \cos \theta$ and thus trace its variation. What values of θ cause the given expression to assume maximum values; minimum?

57. Draw the graph of $\tan \theta + \cot \theta$ and thus trace its variation. What values of θ make the given expression a maximum; a minimum?

58. Draw the graph of arcsin u and trace its variation. [SUGGESTION. Lay off the values of u as abcissas, of arcsin u as ordinates.]

59. Draw the graph of arccos u and trace its variation.

60. Draw the graph of arctan u and trace its variation.

61. Draw the graph of arccot u and trace its variation. What discontinuities are exhibited by the functions of examples 58–61?

62. Find from the table the values of cos 625° 12′; of sin 238° 25′; of tan 324° 6′; of cot 921° 32′.

63. Find without reference to the table the value of

$$\cos 285° \cos 345° + \sin 195° \sin 465°.$$

64. Find without reference to the table the value of

$$\tan 205° \cot 335° + \tan 295° \cot 115°.$$

65. Find all the values between 0 and 2π of

$$\arcsin\left(-\frac{1}{2}\right); \ \arccos\left(-\frac{1}{\sqrt{2}}\right); \ \arctan \frac{1}{3}.$$

66. Find all the values of α between 0 and 2π if

$$\sin 3\alpha = \frac{3}{\sqrt{2}}; \text{ if } \tan 2\alpha = -\sqrt{3}; \text{ if } \cos \frac{\alpha}{2} = \frac{1}{\sqrt{2}}; \text{ if } \cot \frac{\alpha}{3} = -1.$$

67. Find the value of $\sin(2\alpha + \beta)$ if $\alpha = \arcsin \frac{3}{5}$ and of 2d quadrant, $\beta = \arctan \frac{1}{2}$ and of 1st quadrant.

68. Find the value of $\tan(3\alpha + 2\beta)$ if $\alpha = \arcsin \frac{1}{2}$ and of 2d quadrant, $\beta = \arccos \frac{5}{13}$ and of 4th quadrant.

69. Derive the formula for $\sin(\alpha + \beta + \gamma)$ in terms of sine and cosine.

70. Derive the formula for $\cos(\alpha + \beta + \gamma)$ in terms of sine and cosine.

71. Derive the formula for $\tan(\alpha + \beta + \gamma)$ in terms of tangent.

Discuss the results of examples 69–71 in case $\alpha + \beta + \gamma = \pi$.

72. In the results of examples 69–71 put $\beta = \gamma = \alpha$, and thus obtain the formulas for $\sin 3\alpha$, $\cos 3\alpha$, and $\tan 3\alpha$.

73. Find the value of $\cos(3\alpha - 2\beta)$ if $\alpha = \arccos(-\frac{8}{17})$ and of 3d quadrant and $\beta = \arctan\frac{3}{4}$ and of 1st quadrant.

74. Find the value of $\cot(4\alpha - \beta)$ if $\alpha = \arcsin\frac{4}{5}$ and of 1st quadrant and $\beta = \arccos(-\frac{12}{13})$ and of 2d quadrant.

75. Find the value of $\sec(\alpha + \beta)$ if $\alpha = \arctan\frac{5}{12}$ and $\beta = \arcsin\frac{7}{25}$, both of 1st quadrant.

76. Show that $\sin 2 = \dfrac{2\tan\alpha}{1 + \tan^2\alpha}$.

77. Show that $\cos 2\alpha = \dfrac{1 - \tan^2\alpha}{1 + \tan^2\alpha}$.

78. Show that $\tan\left(45° - \dfrac{\alpha}{2}\right) = \csc\alpha + \cot\alpha$.

79. Show that $\cot\left(45° - \dfrac{\alpha}{2}\right) = \csc\alpha - \cot\alpha$.

Transform into products or quotients the following expressions (80–84):

80. $\cot\alpha + \tan\alpha$.

81. $\cot\alpha - \tan\alpha$.

82. $1 + \tan\alpha\tan\beta$.

83. $\cot\alpha - \tan\beta$.

84. $\dfrac{\cot\alpha + \cot\beta}{\tan\alpha + \tan\beta}$.

If $\alpha + \beta + \gamma = \pi$, show that (85–87)

85. $\sin\alpha + \sin\beta + \sin\gamma = 4\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}$. [SUGGESTION. Apply Art. 70 to the first two terms and Art. 67 to the third term.]

86. $\cos\alpha + \cos\beta + \cos\gamma = 1 + 4\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}$.

87. $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$. (See example 71.)

88. How does it appear from example 87 that either all three angles of a triangle are acute or else two are acute and one obtuse? (Consider the signs.)

89. How does it appear from example 87 that if one angle of a triangle is obtuse, it is numerically nearer 90° than either of the acute angles?

In the following equations find the angle:

90. $\tan 2\alpha\tan\alpha = 1$.

91. $\sin(60° - \beta) - \sin(60° + \beta) = \frac{1}{2}$.

92. $\cos 6\gamma - \cos 2\gamma = 0.$

93. $r\sin\theta = 8, r\cos\theta = 15.$

94. $r\sin\theta\cos\phi = 3, r\sin\theta\sin\phi = 4, r\cos\theta = 12.$

95. Show that $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}.$

96. Show that $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15} = \frac{1}{16}.$

97. Show that $2\sin\frac{\alpha}{2} = \pm\sqrt{1+\sin\alpha} \pm \sqrt{1-\sin\alpha}.$

98. Show that $2\cos\frac{\alpha}{2} = \pm\sqrt{1+\sin\alpha} \mp \sqrt{1-\sin\alpha}.$

99. The formula for the horizontal range of a projectile fired at an elevation α with a muzzle speed u, is $\frac{u^2}{g}\sin 2\alpha$. Show that the maximum range is attained for an elevation of 45°.

100. A triangle is formed by two given sides of constant length b and c, including a variable angle α. For what value of α is the third side a maximum; the area a maximum?

SECONDARY TRIGONOMETRIC FUNCTIONS

In addition to the trigonometric functions defined in Art. 6, 32, and 54, there are certain other expressions which are also functions of the angle. While of less importance than the six primary functions, an investigation of their properties will be valuable, not only for the results obtained, but as a review of the fundamental principles of trigonometry.

We may define, then,

$$\text{versed sine } \alpha = \text{vers } \alpha = 1 - \cos\alpha,$$
$$\text{coversed sine } \alpha = \text{covers } \alpha = 1 - \sin\alpha,$$
$$\text{exsecant } \alpha = \text{exsec } \alpha = \sec\alpha - 1,$$
$$\text{excosecant } \alpha = \text{excsc } \alpha = \csc\alpha - 1.$$

101. By reference to Fig. 53, show that

$$\text{vers } \alpha = 1 - \frac{x}{v}, \qquad \text{covers } \alpha = 1 - \frac{y}{v},$$

$$\text{exsec } \alpha = \frac{v}{x} - 1, \qquad \text{excsc } \alpha = \frac{y}{v} - 1.$$

102. By reference to Figs. 60–63, show that, in line representations,

$\text{vers } \alpha = MA,$ $\qquad$ $\text{covers } \alpha = NB,$

$\text{exsec } \alpha = PT,$ $\qquad$ $\text{excsc } \alpha = PS.$

103. Determine the signs and limitations in value of each of the four secondary functions in the different quadrants.

104. Show that if the quadrant of the angle and the value of any one of its ten functions are given, the values of the other nine can be found.

105. Find the values of the four secondary functions, given:

$\alpha = \arcsin\left(-\frac{20}{101}\right)$ and of 3d quadrant; $\beta = \arccos \frac{80}{89}$ and of 4th quadrant;
$\gamma = \arctan\left(-\frac{17}{144}\right)$ and of 2d quadrant; $\delta = \text{arccot}\ \frac{45}{28}$ and of 1st quadrant.

106. Find all ten functions, given:

$\alpha = \text{arcvers}\ \frac{50}{61}$ and of 4th quadrant; $\beta = \text{arccovers}\ \frac{18}{25}$ and of 2d quadrant;
$\gamma = \text{arcexsec}\ \frac{2}{61}$ and of 1st quadrant; $\delta = \text{arcexcsc}\ \frac{25}{8}$ and of 3d quadrant.

107. Trace the variation of each of the secondary functions as the angle varies from 0 to 2π.

108. Draw the graph of each of the secondary functions. What discontinuities, if any, are present.

109. Find the secondary functions of $k \cdot \frac{\pi}{4}$, for $k = 1, 2, \cdots 8$.

110. Find the secondary functions of $k \,.\, \frac{\pi}{6}$, for $k = 1, 2, \cdots 12$.

111. Verify the relations of Art. 61 for the secondary functions.

112. Determine the relations analogous to those of Art. 62 affecting the secondary functions. [SUGGESTION. Use Art. 60.]

113. By means of the cosine series, Art. 77, Eq. (5), show that $\lim\limits_{\theta \doteq 0} \frac{\text{vers}\,\theta}{\theta} = 0$.

114. From the preceding example, show that $\lim\limits_{\theta \doteq 0} \frac{\text{exsec}\,\theta}{\theta} = 0$.

115. Show that

$$\frac{\text{vers}\,\alpha + \text{covers}\,\alpha}{\text{vers}\,\alpha - \text{covers}\,\alpha} = \frac{\text{exsec}\,\alpha + \text{excsc}\,\alpha}{\text{exsec}\,\alpha - \text{excsc}\,\alpha} = \frac{2\,\text{vers}\,\alpha\,\text{covers}\,\alpha}{\text{vers}\,\alpha - \text{covers}\,\alpha}.$$

116. Show that $\text{exsec}^2\beta + 2\,\text{exsec}\,\beta = \tan^2\beta$.

117. Solve and find γ in degrees: $2\,\text{vers}\,\gamma(2 - \text{vers}\,\gamma) = 1$.

118. Solve and find δ in degrees: $\tan^2\delta + \text{exsec}\,\delta = 4$.

119. Show that, if α is the angle at the center of a circle of radius r, the ordinate at the middle of the chord is given by the formula $m = r\,\text{vers}\frac{\alpha}{2}$. Find m for $r = 1433$, $\alpha = 11° 32'$.

120. If τ is the intersection angle of two tangents to a circle of radius r, the shortest distance of their point of intersection from the arc is given by the formula $d = r\,\text{exsec}\frac{\tau}{2}$. Find d for $r = 5730$, $\tau = 5° 32'$.

121. Reduce the first member to the second in the identity

$$(\text{exsec}\,\alpha + \text{vers}\,\alpha)(\text{excsc}\,\alpha + \text{covers}\,\alpha) = \sin\alpha\cos\alpha.$$

122. Show that $\text{vers}\,2\,\alpha = 2\sin^2\alpha$.

123. Show that $\text{exsec}\,2\,\alpha = \frac{2\sin^2\alpha}{1 - 2\sin^2\alpha}$.

124. Show that $\text{excsc}\,2\,\alpha\cos 2\,\alpha = \tan\alpha$.

125. At points in a straight line ordinates are erected such that for each point (x, y), $x = \text{vers}\,(\arcsin y)$. Show that the graph thus determined is a circle tangent to the Y-axis at the origin.

126. At each point in a circular arc the radius is extended an amount equal to the exsecant (in line values) of the arc measured from a fixed point in it. What is the graph thus determined?

127. Two equal circles have their centers in the same horizontal line. Show that the horizontal distance between two points in the neighboring arcs is equal to twice the versed sine (in line values) of the arc, measured from the point of tangency.

128. Two tangents to a circle intersect at an angle τ. Show that the distance of the point of intersection from the midpoint of the chord of contact equals $\text{exsec}\,\tau + \text{vers}\,\tau$ (in line values).

129. Show how the versed sine is of practical use in staking out a circular railroad track passing through three given points.

130. Show how the exsecant is of practical use in staking out a circular railroad spur of given radius branching tangentially from a straight track.

Compute the missing parts of the following triangles, distinguishing right from oblique:

	α	β	γ	a	b	c
131.	37° 42.8′		90°	6244.8		
132.	72° 25.6′		90°			64.863
133.			90°	375.84	296.57	
134.		54° 36.9′		24.465	42.850	
135.		136° 36.8′		36902		37490
136.	68° 51.5′		90°		7532.8	
137.			90°	396.45		531.53
138.			90°	.005428	.006395	
139.	148° 24′			7.4536	5.3648	
140.				.038456	.028638	.051524
141.	125° 34.6′	35° 25.3′				2584.6
142.			24° 36.8′	2.4657	3.6542	
143.		80° 04.5′	90°			30.007
144.			94° 46.8′		34.086	52.475
145.				.93274	.40586	.63208
146.		76° 46.3′	85° 38.7′			8.4637
147.			90°		29846	53857
148.	29° 57.4′	43° 52.6′		64.475		
149.	17° 46.8′				.39475	.29478
150.				36875	28467	48542

151. In a given triangle $a = 280$, $c = 420$, $\gamma = 38°$; find the radius of the circumscribed circle.

152. In a given circle $a = 63$, $b = 81$, $\gamma = 54°$; find the lengths of the bisectors of the interior angles.

153. The sides of a given triangle are 220, 350, 440; find the lengths of the three medians.

154. In a given triangle $b = 340$, $\alpha = 48°$, $\gamma = 63°$; find the lengths of the radii of the inscribed and of the three escribed circles.

155. A boat drifts in a stream whose current runs 4 miles an hour due east under a breeze of 10 miles an hour from the southwest. Determine the motion during 35 minutes, if the resistance reduces the effect of the wind 30%.

156. Three forces of 1800, 2200, and 2700 dynes are in equilibrium; find the angles they make with one another.

157. A helical spring is fastened to the door 16 inches from the axis of the hinges, and to the jamb 4 inches from the same line in the same horizontal plane. Find the length of the spring when the door is closed, open at 30°, 45°, 70°, 90°, 120°. Neglect the thickness of the door.

158. A cable 30 feet long is suspended from the tops of two vertical poles 20 feet apart and 15 and 18 feet high, and bears a load of 200 pounds hanging from it by a trolley. Find the position of the trolley when at rest, and the lengths, inclinations to the horizon, and (common) tensions of the segments of the cable. Neglect the weight of the cable.

159. Let the data be as in the preceding example, save that the load hangs from a ring knotted at the center of the cable; find the inclinations to the horizon and the (unequal) tensions of the segments. Solve when the ring is knotted at a point 12 feet from the lower end of the cable.

160. The eye is 40 inches in front of a mirror and an object appears to be 35 inches back of it, while the line of sight makes an angle of 48° with the mirror. Find the distance and direction of the object from the eye. (NOTE. The angles of incidence and reflection are equal.)

161. The line from the eye to the object recedes from the mirror at an angle of 32°, while the object is 36 inches from the eye and 12 inches from the mirror. Find the angles of incidence and reflection, and the point of reflection.

162. Two railway tracks intersect at an angle of 75°, and are connected by a circular "Y" of 800 feet radius lying in the obtuse angle and tangent to the two tracks. Find the distances of the points of tangency from the crossing and the length of the "Y".

163. Two railway tracks, intersecting at an angle of 62°, are joined by a circular "Y" in the acute angle and tangent to the two tracks at points 900 feet from the crossing. Find the radius of the "Y" and its length.

164. In setting a door frame 6 feet wide and 8 feet high, the vertical side is found to be 2 inches (horizontally) out of plumb. Find the angles of the parallelogram and the lengths of the diagonals. Is the diagonal of the true rectangle the average (arithmetic mean) of these two?

165. The staking out of a certain building requires the setting of stakes at the vertices of a rectangle 32 x 48 feet. A test of the trial setting shows the figure to be a parallelogram whose sides are as given above, but whose diagonals differ by 9 inches. Find the angle through which the longer sides must be swung to correct distortion, and the chord of the arc through which the back corners must be moved.

166. A triangular roof truss is 80 feet long and divided into 8 equal segments, by vertical members, as shown in Fig. 72. The height being 25 feet, what are the lengths and inclinations to the horizon of the various members?

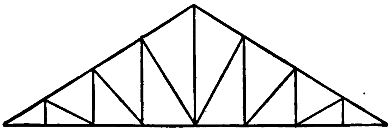

FIG. 72.

167. The triangular roof truss shown in Fig. 73 is 60 feet long and 20 feet high. The bottom chord and rafters are divided into equal segments. Find the lengths and inclinations of the members.

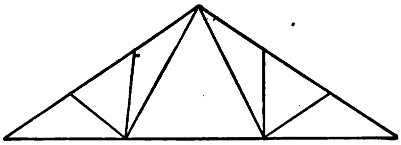

FIG. 73.

168. In order to determine the exact location of the point G (Fig. 74), a base line AB is laid off due north and south, measuring precisely 130 rods. Convenient intermediate stations are chosen, and angles measured as follows: $ABC = 36^\circ\ 35'$, $BAC = 61^\circ\ 10'$, $BAD = 43^\circ\ 54'$, $CAD = 17^\circ\ 16'$, $DCE = 66^\circ\ 36'$, $CDE = 47^\circ\ 41'$, $EDF = 55^\circ\ 48'$, $DEF = 73^\circ\ 12'$, $FEG = 58^\circ\ 32'$, $EFG = 70^\circ\ 28'$. Compute the distances composing the sides of the triangles in the figure.

169. By projecting the distances AC, CE, EG (Fig. 74), perpendicular and parallel to AB, compute the easterly and southerly distances of G from A; find thence the direct distance and direction of G from A.

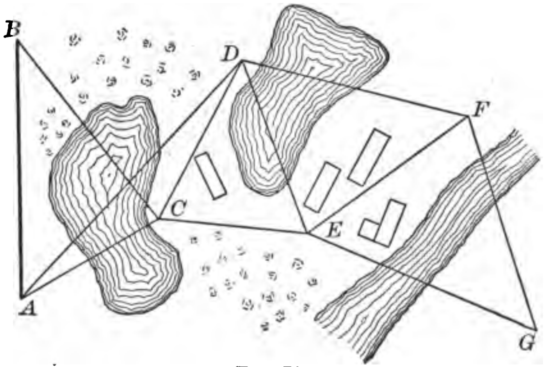

FIG. 74.

TRIGONOMETRY

PART II

SPHERICAL TRIGONOMETRY

CHAPTER X

GEOMETRIC NOTIONS

82. General statement. Spherical trigonometry is merely the application of trigonometry to the determination of the relations between the sides and angles of a spherical triangle, or, as will be seen, between the face angles and dihedrals of a trihedral. By means of the formulas so derived, when three parts are given, the remaining parts can in general be computed.

83. Review of geometric theorems. Three great circle arcs on the surface of a sphere, if not concurrent, form a spherical triangle. If these three great circles be completed, eight triangles will be formed, each of which definitely determines the other seven. In at least one of the eight each side is less than a semicircumference and each angle is less than a straight angle. Hence only those triangles are considered in which each part is less than 180°.

The following theorems, proved in elementary geometry, are restated without proof:

The sum of any two sides of a spherical triangle is greater than the third side.

The sum of the three sides of a spherical triangle is less than 360°. The triangle may have one, two, or three sides greater than 90°.

The sum of the three angles of a spherical triangle is greater than 180°, and less than 540°. The triangle may have one, two, or three angles greater than 90°.

If two angles of a spherical triangle are unequal, the opposite sides are unequal, and the greater side is opposite the greater angle.

If the vertices of a spherical triangle are the poles of the sides of a second triangle, the vertices of the second are also the poles of the sides of the first. The two triangles are then said to be polar to each other. In two polar triangles, each angle of either is the supplement of the opposite side of the other.

Two spherical triangles are equal, or symmetrical, if in the two the following three parts of the one are equal to the corresponding parts of the other:

Three sides;
Three angles;
Two sides and the included angle;
Two angles and the included side.

A spherical triangle is therefore definitely determined, and the remaining parts can be found by construction or computation, if there are given the three sides, or the three angles, or two sides and the included angle, or two angles and the included side. If two sides and the angle opposite one of them, or two angles and the side opposite one, are given, an ambiguity arises analogous to that in case of plane triangles. The formulas for the computation of the unknown parts in each case are derived in the following chapters.

Each of the foregoing statements should be verified by careful construction on a spherical blackboard.

If the vertices of a spherical triangle are joined to the center of the sphere, these radii form the edges of a trihedral whose face angles are measured by the sides of the triangle, and whose dihedral angles are measured by the angles of the triangle. Each of the theorems just stated has its counterpart with regard to the trihedral. If the vertices of two polar triangles are joined to the center of the sphere, the edges of each trihedral so formed are respectively perpendicular to the faces of the other.

Each of the formulas to be derived may be applied directly to the solution of trihedrals.

CHAPTER XI

RIGHT SPHERICAL TRIANGLES

84. Formulas for solution. Let ABC, in Fig. 75, be a spherical triangle, in which the angle at C is a right angle, while each of the remaining parts is less than 90°. Denote the angles at A, B, C, by α, β, γ, respectively, and the sides opposite them by a, b, c. Join the vertices to the center of the sphere. Then in the trihedral O–ABC the face angles COB, AOC, BOA are equal to a, b, c, respectively, and the dihedrals B–OA–C, C–OB–A, A–OC–B are respectively equal to α, β, γ.

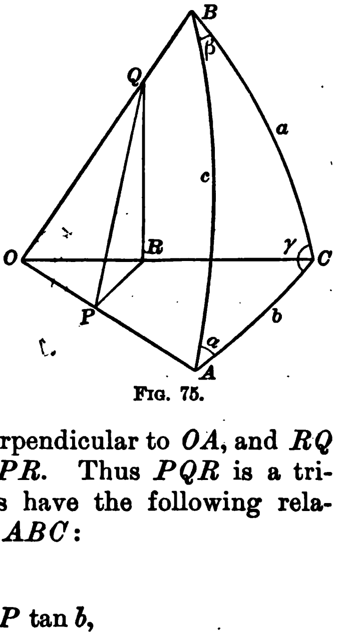

FIG. 75.

Through any point P in OA pass a plane perpendicular to the radius OA. Of its traces in the faces of the trihedral, PQ and PR are both perpendicular to OA, and RQ is perpendicular to OC, and hence to PR. Thus PQR is a triangle right-angled at R, and its parts have the following relations to those of the spherical triangle ABC:

$$RPQ = \alpha,$$

$$PR = OR \sin b = OP \tan b,$$

$$RQ = OQ \sin a = OR \tan a,$$

$$PQ = OQ \sin c = OP \tan c.$$

Applying the formulas of Art. 16 to the triangle PQR, and substituting the values thus obtained, we derive the following formulas:

$$RQ = PQ \sin RPQ,$$

or,

$$OQ \sin a = OQ \sin c \sin \alpha.$$

Dividing by OQ, we have

(1) $$\sin a = \sin c \sin \alpha.$$

The parts b and B may evidently be substituted for a and A, whence

(2) $$\sin b = \sin c \sin \beta.$$

$$PR = PQ \cos RPQ,$$

or, $$OP \tan b = OP \tan c \cos \alpha.$$

Dividing by OP, we have

(3) $$\tan b = \tan c \cos \alpha,$$

whence, also,

(4) $$\tan a = \tan c \cos \beta.$$

$$RQ = PR \tan RPQ,$$

or, $$OR \tan a = OR \sin b \tan \alpha.$$

Dividing by OR, we have

(5) $$\tan a = \sin b \tan \alpha,$$

whence, also,

(6) $$\tan b = \sin a \tan \beta.$$

We also obtain by multiplication

$$OP \cdot OQ \cdot OR \sin a \sin b \tan c = OP \cdot OQ \cdot OR \tan a \tan b \sin c.$$

Dividing by $OP \cdot OQ \cdot OR$, and also by $\tan a \tan b \tan c$, we have

(7) $$\cos c = \cos a \cos b.$$

Multiplying (1) and (7), and dividing by (4), member for member, we obtain

$$\frac{\sin a \cos c}{\tan a} = \frac{\sin c \sin \alpha \cos a \cos b}{\tan c \cos \beta}.$$

Canceling and clearing of fractions, we have

(8) $$\cos \beta = \cos b \sin \alpha.$$

Similarly from (2), (7), and (3), we obtain

(9) $$\cos \alpha = \cos a \sin \beta.$$

Multiplying (8) and (9), and dividing by (7), we get

$$\frac{\cos\alpha\cos\beta}{\cos c} = \frac{\cos a\cos b\sin\alpha\sin\beta}{\cos a\cos b},$$

and, on simplifying,

$$\cos c = \cot\alpha\cot\beta. \tag{10}$$

85. Extension of formulas. The preceding demonstration is based upon the assumption that both a and b are less than 90°. It remains to be shown that the resulting formulas also hold in case one or both these sides are greater than 90°.

Suppose $b > 90°$. Extend the sides AB and AC so as to complete the lune, as in Fig. 76.

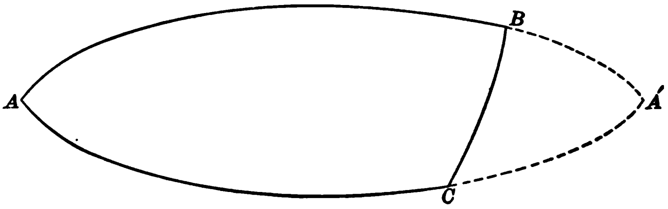

Fig. 76.

We have then the triangle $A'BC$, right-angled at C, and with $b' < 90°$. Moreover,

$$a' = a,\ b' = 180° - b,\ c' = 180° - c,$$
$$\alpha' = \alpha,\ \beta' = 180° - \beta,\ \gamma' = \gamma = 90°.$$

The formulas (1) to (10) of Art. 84 hold for a', b', c', α', β', γ'. Substituting in these formulas in turn the values of these parts in terms of a, b, c, α, β, γ, and making use of the relations of Art. 37, the formulas will be reproduced identically for $a, \cdots, \gamma$.

Again, let $a > 90°$, $b > 90°$. Extend the sides CA and CB so

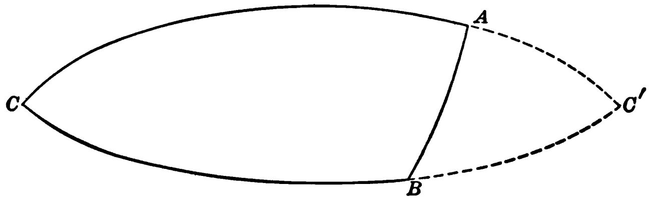

Fig. 77.

as to complete the lune, as in Fig. 77. The triangle ABC' is right-angled at C', and has $a' < 90°$, $b' < 90°$. Further,

$$a' = 180° - a, \quad b' = 180° - b, \quad c' = c,$$
$$\alpha' = 180° - \alpha, \quad \beta' = 180° - \beta, \quad \gamma' = \gamma = 90°.$$

Formulas (1) to (10) hold for $a', \cdots, \gamma'$, and proceeding as before, they are identically reproduced for $a, \cdots, \gamma$.

86. Napier's rules of circular parts. The memory of these formulas is assisted by their analogy to the formulas for plane right triangles. Thus we may write

$$(1) \quad \sin\alpha = \frac{\sin a}{\sin c}, \qquad (2) \quad \sin\beta = \frac{\sin b}{\sin c},$$

$$(3) \quad \cos\alpha = \frac{\tan b}{\tan c}, \qquad (4) \quad \cos\beta = \frac{\tan a}{\tan c},$$

$$(5) \quad \tan\alpha = \frac{\tan a}{\sin b}, \qquad (6) \quad \tan\beta = \frac{\tan b}{\sin a}.$$

Some prefer to memorize the ten formulas. Others make use of the following two Rules of Circular Parts, due to Lord John Napier. Omitting the right angle γ, arrange the five remaining parts in their natural order, using, however, instead of c, α, β, their supplements, as in Fig. 78.

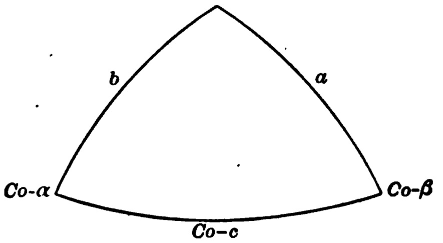

FIG. 78.

If any one of the five parts is taken as the *middle* part, the two immediately adjoining are called the *adjacent* parts, and the two remaining the *opposite* parts. *E.g.* if a is the middle part, co-c and co-β are the adjacent parts, b and co-α the opposite parts.

Napier's rules are the following:

I. The sine of the middle part is equal to the product of the tangents of the two adjacent parts;

II. The sine of the middle part is equal to the product of the cosines of the two opposite parts.

The alliterative element in the device is obvious.

The validity of the rules appears only from the fact that, when each is applied in the five possible ways, formulas (1) to (10) of Art. 84 are obtained. The student should verify this in detail.

87. Limitations as to quadrants. Each part of the triangle may be either less or greater than 90°, and it will be recalled that, while the sine of any part is always positive, the cosine and tangent are positive or negative according as the part is less or greater than 90°. Observance of the algebraic law of signs will enable us to determine the quadrant of any part which is found by means of its cosine or tangent. When the part is found from its sine, an apparent ambiguity arises. This is cleared up, however, by consideration of certain limitations that follow from some of the formulas of Art. 84.

In formula (5), $\tan a = \sin b \tan \alpha$, $\sin b$ is always positive. Hence $\tan a$ and $\tan \alpha$ have the same sign. It follows that a and α are always of the same quadrant. Moreover, since $\sin b < 1$, $\tan a < \tan \alpha$. Therefore α lies nearer 90° than does a; *i.e.* when a and α are both of the first quadrant, $a < \alpha$; when they are both of second quadrant, $a > \alpha$.

From formula (6) the same conclusions may be drawn concerning b and β.

Again, from formula (7), $\cos c = \cos a \cos b$, it results that when any one of the three sides is less than 90°, the other two are of the same quadrant; but when one side is greater than 90°, the other two are of different quadrants. Since the cosine is numerically less than unity, $\cos c$ is numerically less than $\cos a$ or $\cos b$. Hence, c is nearer 90° than either a or b.

From (8) it follows that $\cos \beta$ is numerically less than $\sin \alpha$.

If $\alpha < 90°$, $\beta < 90°$,

$$\cos \beta < \cos (90° - \alpha).$$

Whence, $$\beta > 90° - \alpha,$$

or $$\alpha + \beta > 90°.$$

This relation also follows from the fact that in any spherical triangle $\alpha + \beta + \gamma > 180°$.

If $\alpha > 90°$, $\beta < 90°$,

$$\cos \beta < \sin (180° - \alpha),$$

or $$\cos \beta < \cos (\alpha - 90°).$$

Whence, $$\beta > \alpha - 90°,$$

or $$\alpha - \beta < 90°.$$

If $\alpha > 90^\circ$, $\beta > 90^\circ$, in which case $\cos b$ is negative,

$$\cos(180^\circ - \beta) < \sin(180^\circ - \alpha),$$

or

$$\cos(180^\circ - \beta) < \cos(\alpha - 90^\circ).$$

Whence,

$$180^\circ - \beta > \alpha - 90^\circ,$$

or

$$\alpha + \beta < 270^\circ.$$

There is only one solution possible in all cases except when the given parts are a side and its opposite angle. In this case there are two solutions. The two triangles are colunar, and the remaining parts of the one are the supplements of the corresponding parts of the other. This appears readily from inspection of Fig. 76.

88. Solution of right spherical triangles. When two parts of a right spherical triangle are given, the remaining parts may be computed by means of the three formulas so selected from the ten of Art. 84 that each contains the two given parts together with one of the remaining parts. The check is afforded by means of the formula involving the three unknown parts.

Logarithmic computation is merely a method of obtaining arithmetical products and quotients, without regard to their signs. The sign of the result must be independently ascertained from consideration of the signs of the factors. The limitations mentioned in Art. 87 should be carefully observed.

EXAMPLE 1. Given $a = 33^\circ\,18.5'$, $c = 69^\circ\,24.4'$.

The solution is effected by formulas (1), (4), and (7), while (8) affords the check. By Art. 87, it appears that b, α, β, must all be less than 90°. The work may be compactly arranged as follows:

$$\begin{array}{ll}
a = 33^\circ\,18.5' & \log\sin 9.73969 - 10 \\
c = 69^\circ\,24.4' & \text{colog}\sin 0.02868 \\
\hline
\alpha = 35^\circ\,55.1' & \log\sin 9.76837 - 10 \\
\beta = 75^\circ\,42.4' & \\
b = 65^\circ\,06.7' &
\end{array}$$

$$\begin{array}{lll}
\log\tan 9.81762 - 10 & \text{colog}\cos 0.07793 & \\
\log\cot 9.57489 - 10 & \log\cos 9.54621 - 10 & \\
 & & \text{colog}\sin 0.23163 \\
\cline{1-1}
\log\cos 9.39251 - 10 & & \log\cos 9.39251 - 10 \\
\cline{2-3}
 & \log\cos 9.62414 - 10 & \log\cos 9.62414 - 10
\end{array}$$

EXAMPLE 2. Given $b = 145^\circ\,42.5'$, $\beta = 128^\circ\,31.7'$.

The solution is effected by formulas (2), (6), and (8), and the check by (1). Note that the check formula involves only the sine. One solution has $a < 90°$, $c > 90°$, $\alpha < 90°$; the second solution has $a' > 90°$, $c' < 90°$, $\alpha' > 90°$; $a + a' = c + c' = \alpha + \alpha' = 180°$.

$$\begin{array}{rl|l}
b = & 145° \; 42.5' & \log\tan 9.83374 - 10 \\
\beta = & 128° \; 31.7' & \log\cot 9.90104 - 10 \\
\hline
a = & 32° \; 53.2' & \log\sin 9.73478 - 10 \\
c = & 133° \; 55.8' & \\
\alpha = & 48° \; 56.1' &
\end{array}$$

$$\begin{array}{lll}
\log\sin 9.75082-10 & \text{colog}\cos 0.08293 & \\
\text{colog}\sin 0.10663 & \log\cos 9.79442-10 & \\
 & & \log\sin 9.73478-10 \\
\hline
\log\sin 9.85745-10 & & \text{colog}\sin 0.14255 \\
 & \overline{\log\sin 9.87735-10} & \overline{\log\sin 9.87733-10}
\end{array}$$

In the second triangle, $a = 147° \; 06.8'$, $c = 46° \; 04.2'$, $\alpha = 131° \; 03.9'$.

EXERCISE XXIV

Find the missing parts of the following spherical triangles, in which $\gamma = 90°$.

	α	β	a	b	c
1.			39° 42.2′		79° 35.3′
2.	72° 14.3′			29° 17.2′	
3.		80° 10.2′			53° 07.1′
4.	78° 13.9′		64° 28.3′		
5.	65° 17.2′	153° 22.5′			
6.			49° 07.7′	100° 10.1′	
7.		127° 42.6′	83° 24.2′		
8.				163° 12.3′	81° 43.5′
9.		58° 17.5′			58° 17.5′
10.	132° 23.1′	77° 11.3′			
11.		112° 44.5′	61° 45.4′		
12.				57° 53.2′	121° 32.6′
13.		99° 08.7′		131° 06.4′	
14.	75° 15.6′				129° 16.8′
15.			143° 32.8′	78° 15.5′	
16.	145° 43.2′		133° 17.1′		

89. Quadrantal and isosceles triangles. If one side of a spherical triangle is 90°, its polar triangle is right-angled and may be solved as in the preceding article. The supplements of the sides and angles so found are the respective angles and sides of the given triangle, in accordance with Art. 83.

If two sides of a spherical triangle are equal, the great circle arc drawn from the vertex of the included angle perpendicular to the opposite side divides the triangle into two symmetrical right triangles, which may be solved as in the preceding article and the unknown parts of the given triangle found.

EXAMPLE. Given $a = 54° 47.4$, $b = 107' 24.7'$, $c = 90°$.

In the polar triangle, $\alpha' = 125° 12.6'$, $\beta' = 72° 35.3'$, $\gamma' = 90°$. The solution of this triangle is effected by formulas (8), (9), and (10), and the check by (7). By Art. 87, $a' > 90°$, $b' < 90°$, $c' > 90°$.

$$\begin{array}{ll}
\alpha' = 125° \, 12.6' & \log\cos 9.76086 - 10 \\
\underline{\beta' = 72° \, 35.3'} & \underline{\text{colog}\sin 0.02037 } \\
a' = 127° \, 10.6' & \log\cos 9.78123 - 10 \\
b' = 68° \, 31.0' & \\
c' = 103° \, 05.3' &
\end{array}$$

$$\begin{array}{lll}
\text{colog}\sin 0.08775 & \log\cot 9.84861 - 10 & \\
\log\cos 9.47601 - 10 & \log\cot 9.49639 - 10 & \\
\hline
& & \log\cos 9.78123 - 10 \\
\log\cos 9.56376 - 10 & & \underline{\log\cos 9.56376 - 10} \\
\cline{2-2}
& \log\cos 9.34500 - 10 & \log\cos 9.34499 - 10
\end{array}$$

Then $\alpha = 52° 49.4'$, $\beta = 111° 29.0'$, $\gamma = 76° 54.7'$.

EXERCISE XXV

1. What limitations upon the parts of a quadrantal triangle may be deduced from the various conclusions of Art. 87?

2. From the formulas of Art. 84, derive the following direct relations between the parts of an isosceles triangle, in which α denotes one of the equal angles, β the unequal angle, a one of the equal sides, and b the unequal side:

$$\sin \tfrac{1}{2} b = \sin a \sin \tfrac{1}{2} \beta, \qquad \cos \tfrac{1}{2} \beta = \cos \tfrac{1}{2} b \sin \alpha,$$
$$\tan \tfrac{1}{2} b = \tan a \cos \alpha, \qquad \cos a = \cot \alpha \cot \tfrac{1}{2} \beta.$$

3. Derive the following relation between the side and angle of an equilateral triangle: $2 \sin \frac{1}{2} \alpha \cos \frac{1}{2} a = 1$.

Find the missing parts of the following triangles.

	α	β	γ	a	b	c
4.				90°	90°	90°
5.		61° 38.9′		156° 19.8′		90°
6.	120° 21.5′	131° 35.7′				90°
7.			110° 48.4′	63° 25.4′	63° 25.4′	
8.		106° 55.5′	93° 47.8′			90°
9.			118° 56.6′	130° 31.8′		90°
10.				113° 16.2′	86° 21.5′	113° 16.2′
11.				72° 16.3′	54° 06.4′	90°
12.	112° 16.2′	143° 20.3′				90°
13.	72° 33.6′	102° 15.2′	102° 15.2′			
14.		71° 13.2′			52° 12.6′	52° 12.6′
15.				35° 12.4′	35° 12.4′	35° 12.4′
16.	132° 16.6′	132° 16.6′	132° 16.6′			
17.	27° 14.5′	27° 14.5′	27° 14.5′			

CHAPTER XII

OBLIQUE SPHERICAL TRIANGLES

90. Law of sines. In the spherical triangle ABC, denote the angles at A, B, C, by α, β, γ, and the opposite sides by a, b, c, respectively. Let the sides a and b be less than 90°. Join the vertices to the center of the sphere, as in Fig. 79. Through any point P in OC pass a plane perpendicular to OA and a second plane perpendicular to OB. The traces RT and RP of the first plane in the faces of the trihedral are perpendicular to OA, and the traces ST and SP of the second plane are perpendicular to OB. The line of intersection PT of the two planes is perpendicular to the face AOB, and hence to RT and ST. Thus the angles

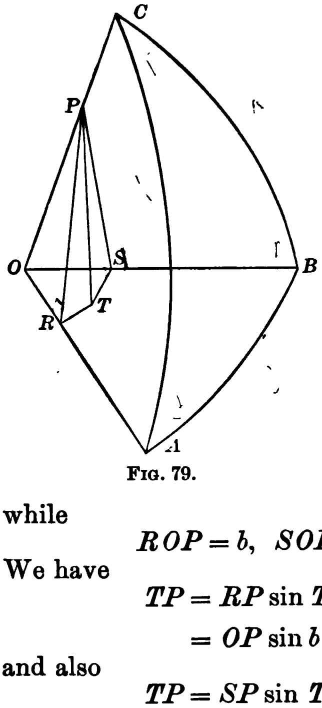

Fig. 79.

$$ORP = OSP = TRP = TSP = 90°,$$

while

$$ROP = b, \quad SOP = a, \quad TRP = \alpha, \quad TSP = \beta.$$

We have

$$\begin{aligned} TP &= RP \sin TRP = OP \sin ROP \sin TRP \\ &= OP \sin b \sin \alpha, \end{aligned}$$

and also

$$\begin{aligned} TP &= SP \sin TSP = OP \sin SOP \sin TSP \\ &= OP \sin a \sin \beta. \end{aligned}$$

Equating the values of TP, and dividing by OP, we obtain

$$\sin a \sin \beta = \sin b \sin \alpha,$$

or

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

91. Laws of cosines. (1) **Law of cosines for the sides.** Let the sides a and b be less than 90°. Join the vertices to the center of the sphere, as in Fig. 80, and through any point P in OC pass a plane perpendicular to OC. Then

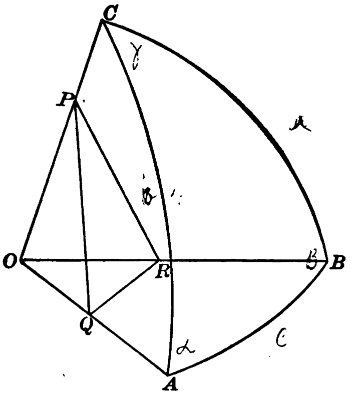

FIG. 80.

$$OPQ = OPR = 90°,$$

while

$$ROP = a,\quad QOP = b,$$

$$QOR = c,\quad QPR = \gamma.$$

We have

$$QR^2 = OQ^2 + OR^2 - 2\,OQ \cdot OR \cos c,$$

and also

$$QR^2 = PQ^2 + PR^2 - 2\,PQ \cdot PR \cos \gamma.$$

Subtracting, we get

$$0 = OQ^2 - PQ^2 + OR^2 - PR^2 - 2\,OQ \cdot OR \cos c + 2\,PQ \cdot PR \cos \gamma.$$

But

$$OQ^2 - PQ^2 = OR^2 - PR^2 = OP^2$$

and

$$OQ = OP \sec b,\quad PQ = OP \tan b,\quad OR = OP \sec a,\quad PR = OP \tan a.$$

Introducing these values, we have

$$0 = 2\,OP^2 - 2\,OP^2 \sec a \sec b \cos c + 2\,OP^2 \tan a \tan b \cos \gamma;$$

whence, we obtain

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma,$$

and similarly,

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha,$$

$$\cos b = \cos a \cos c + \sin a \sin c \cos \beta.$$

(2) **Law of cosines for the angles.** Passing to the polar triangle, and suppressing the accents, we obtain the formulas,

$$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c,$$

$$\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a,$$

$$\cos \beta = -\cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos b.$$

The formulas of this article are fundamental, but are not adapted to the use of logarithms. They may also be written in the form

$$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \text{ etc.,}$$

$$\cos a = \frac{\cos \alpha - \cos \beta \cos \gamma}{\sin \beta \sin \gamma}, \text{ etc.}$$

92. Extension of the formulas. In Arts. 90 and 91, it was assumed that both a and b were less than 90°. It remains to be shown that the formulas derived hold when either or both are greater than 90°.

Suppose $a > 90°$, $b < 90°$. Extend the sides BA and BC so as to complete the lune, as in Fig. 81. In the colunar triangle $AB'C$, a' and b' are both less than 90°, and the formulas of Arts. 90 and 91 hold. But $b' = b$, and $\beta' = \beta$, while the remaining parts are supplementary to the corresponding parts of ABC. Upon making the proper substitutions, the formulas are identically reproduced.

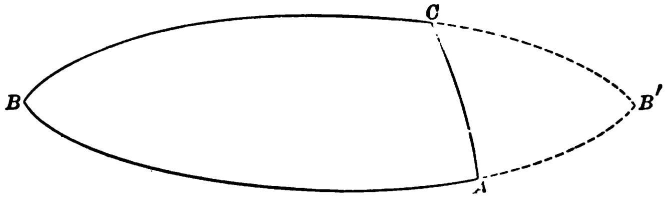

FIG. 81.

Again, let $a > 90°$, $b > 90°$. Extend the sides CA and CB so as to complete the lune, as in Fig. 82. In the colunar triangle ABC', both a' and b' are less than 90°, and the formulas of Arts. 90 and 91 hold. Substituting, the formulas are reproduced as before.

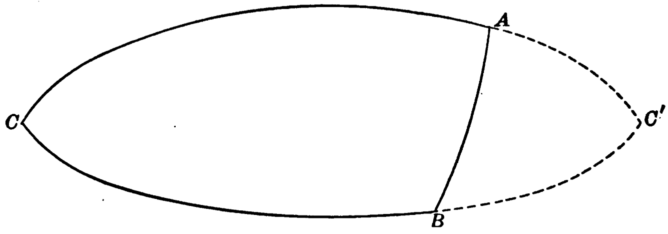

FIG. 82.

93. Angles in terms of the sides. From Art. 91 we have

$$\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2}=\frac{\cos b\cos c+\sin b\sin c-\cos a}{2\sin b\sin c}$$

$$=\frac{\cos(b-c)-\cos a}{2\sin b\sin c}$$

$$=\frac{\sin\frac{1}{2}(a-b+c)\sin\frac{1}{2}(a+b-c)}{\sin b\sin c}.$$

Again,

$$\cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2}=\frac{\cos a-\cos b\cos c+\sin b\sin c}{2\sin b\sin c}$$

$$=\frac{\cos a-\cos(b+c)}{2\sin b\sin c}$$

$$=\frac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a+b+c)}{\sin b\sin c}.$$

Putting

$s=\frac{1}{2}(a+b+c)$, we obtain

$$\sin\frac{\alpha}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}},$$

$$\cos\frac{\alpha}{2}=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}},$$

$$\tan\frac{\alpha}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}.$$

The third formula may be written in the form

$$\tan\frac{\alpha}{2}=\frac{1}{\sin(s-a)}\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

or, putting

$$\tan r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$\tan\frac{\alpha}{2}=\frac{\tan r}{\sin(s-a)}.$$

There are corresponding formulas for $\frac{\beta}{2}$ and $\frac{\gamma}{2}$. Since the half angles are less than 90°, the positive square roots are to be taken.

By reasoning exactly like that of Art. 44, r is seen to be the spherical radius of the small circle inscribed in the triangle ABC.

94. Sides in terms of the angles. Precisely as in Art. 93, we obtain the following formulas, in which $S = \frac{1}{2}(\alpha + \beta + \gamma)$.

$$\sin\frac{a}{2} = \sqrt{\frac{-\cos S \cos(S-\alpha)}{\sin\beta \sin\gamma}},$$

$$\cos\frac{a}{2} = \sqrt{\frac{(\cos S-\beta)\cos(S-\gamma)}{\sin\beta\sin\gamma}},$$

$$\tan\frac{a}{2} = \sqrt{\frac{-\cos S\cos(S-\alpha)}{\cos(S-\beta)\cos(S-\gamma)}}.$$

As before, we may write

$$\tan\frac{a}{2} = \frac{\cos(S-\alpha)}{\sqrt{\frac{\cos(S-\alpha)\cos(S-\beta)\cos(S-\gamma)}{-\cos S}}},$$

or, putting

$$\cot R = \sqrt{\frac{\cos(S-\alpha)\cos(S-\beta)\cos(S-\gamma)}{-\cos S}},$$

$$\tan\frac{a}{2} = \frac{\cos(S-\alpha)}{\cot R}.$$

There are similar formulas for $\frac{b}{2}$ and $\frac{c}{2}$. As in the preceding article, the positive square is to be taken. The quantity under the radical sign will always be found to be positive when $\alpha + \beta + \gamma$ lies between 180° and 540°.

It may readily be shown that R is the spherical radius of the small circle circumscribed about the triangle ABC.

95. Napier's analogies. Dividing $\tan\frac{a}{2}$ by $\tan\frac{b}{2}$, as given in Art. 94, we get

$$\frac{\tan\frac{a}{2}}{\tan\frac{b}{2}} = \frac{\cos(S-\alpha)}{\cos(S-\beta)}.$$

Then

$$\frac{\tan\frac{a}{2} - \tan\frac{b}{2}}{\tan\frac{a}{2} + \tan\frac{b}{2}} = \frac{\cos(S-\alpha) - \cos(S-\beta)}{\cos(S-\alpha) + \cos(S-\beta)},$$

whence,

$$\frac{\sin\frac{a}{2}\cos\frac{b}{2}-\cos\frac{a}{2}\sin\frac{b}{2}}{\sin\frac{a}{2}\cos\frac{b}{2}+\cos\frac{a}{2}\sin\frac{b}{2}}=\frac{2\sin\frac{1}{2}(2S-\alpha-\beta)\sin\frac{\alpha-\beta}{2}}{2\cos\frac{1}{2}(2S-\alpha-\beta)\cos\frac{\alpha-\beta}{2}},$$

or

$$\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}=\tan\frac{\gamma}{2}\tan\frac{\alpha-\beta}{2},$$

or

$$\tan\frac{\alpha-\beta}{2}=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{\gamma}{2}.$$

Multiplying $\tan\frac{a}{2}$ by $\tan\frac{b}{2}$, we get

$$\tan\frac{a}{2}\tan\frac{b}{2}=\frac{\sin\frac{a}{2}\sin\frac{b}{2}}{\cos\frac{a}{2}\cos\frac{b}{2}}=\frac{-\cos S}{\cos(S-\gamma)}.$$

Then

$$\frac{\cos\frac{a}{2}\cos\frac{b}{2}+\sin\frac{a}{2}\sin\frac{b}{2}}{\cos\frac{a}{2}\cos\frac{b}{2}-\sin\frac{a}{2}\sin\frac{b}{2}}=\frac{\cos(S-\gamma)-\cos S}{\cos(S-\gamma)+\cos S},$$

whence,

$$\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}=\frac{\sin\frac{2S-\gamma}{2}\sin\frac{\gamma}{2}}{\cos\frac{2S-\gamma}{2}\cos\frac{\gamma}{2}}$$

$$=\tan\frac{\alpha+\beta}{2}\tan\frac{\gamma}{2},$$

or

$$\tan\frac{\alpha+\beta}{2}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{\gamma}{2}.$$

Combining in like manner the values of $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ given in Art. 93, we obtain the formulas

$$\tan\frac{a-b}{2}=\frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)}\tan\frac{c}{2},$$

$$\tan\frac{a+b}{2}=\frac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}\tan\frac{c}{2}.$$

96. Gauss's formulas. Substituting from Art. 93, and simplifying, we get

$$\sin\frac{\alpha+\beta}{2}=\sin\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\frac{\alpha}{2}\sin\frac{\beta}{2}$$

$$=\frac{\sin(s-b)+\sin(s-a)}{\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}$$

$$=\frac{\sin\frac{1}{2}(2s-a-b)\cos\frac{a-b}{2}}{\sin\frac{c}{2}\cos\frac{c}{2}}\cos\frac{\gamma}{2},$$

or

$$\sin\frac{\alpha+\beta}{2}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{c}{2}}\cos\frac{\gamma}{2}.$$

In the same manner,

$$\sin\frac{\alpha-\beta}{2}=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{c}{2}}\cos\frac{\gamma}{2},$$

$$\cos\frac{\alpha+\beta}{2}=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{c}{2}}\sin\frac{\gamma}{2},$$

$$\cos\frac{\alpha+\beta}{2}=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{c}{2}}\sin\frac{\gamma}{2}.$$

97. Area of a spherical triangle. If E denotes the spherical excess of a spherical triangle, measured in degrees, so that

$$E=\alpha+\beta+\gamma-180°,$$

it has been shown in geometry that the area of the triangle is given by the formula

$$\text{Area}=\frac{\pi r^2E}{180},$$

where r denotes the radius of the sphere.

Thus the area may be found at once when the three angles are known. When other parts are given, the angles may be computed, and the area found. Several convenient formulas have been derived for computing the spherical excess directly from given parts. One of the most advantageous is the following, known as L'Huillier's formula.

$$\begin{aligned}
\tan \tfrac{1}{4} E &= \frac{\sin \frac{1}{4}(\alpha+\beta+\gamma-180^\circ)}{\cos \frac{1}{4}(\alpha+\beta+\gamma-180^\circ)} \\
&= \frac{\sin \frac{1}{2}(\alpha+\beta)-\sin \frac{1}{2}(180^\circ-\gamma)}{\cos \frac{1}{2}(\alpha+\beta)+\cos \frac{1}{2}(180^\circ-\gamma)} \\
&= \frac{\sin \frac{1}{2}(\alpha+\beta)-\cos \frac{1}{2}\gamma}{\cos \frac{1}{2}(\alpha+\beta)+\sin \frac{1}{2}\gamma} \\
&= \frac{\cos \frac{1}{2}(a-b)-\cos \frac{1}{2}c}{\cos \frac{1}{2}(a+b)+\cos \frac{1}{2}c} \cdot \frac{\cos \frac{1}{2}\gamma}{\sin \frac{1}{2}\gamma} \\
&= \frac{\sin \frac{1}{2}(s-b)\sin \frac{1}{2}(s-a)}{\cos \frac{1}{2}s \cos \frac{1}{2}(s-c)} \sqrt{\frac{\sin s \sin (s-c)}{\sin (s-a)\sin (s-b)}} \\
&= \sqrt{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)}.
\end{aligned}$$

98. Solution of oblique spherical triangles. When any three parts of a spherical triangle are given, the three remaining parts may be computed by means of the formulas just derived. Six cases present themselves, according as there are given:

I. Three sides;
II. Three angles;
III. Two sides and the included angle;
IV. Two angles and the included side;
V. Two sides and the angle opposite one;
VI. Two angles and the side opposite one.

In the first four cases the problem is determinate; but in cases V and VI an ambiguity arises similar to that in case of plane triangles. When parts are found from their cosines or tangents, the algebraic signs will determine the quadrant; when found from the sine, the part may be either first or second quadrant.

In addition to the limitations resulting from the theorems stated in Art. 83, certain others follow from the formulas of Arts. 90 to 95.

In the formula of Art. 91,

$$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

the denominator is always positive. Unless a lies nearer 90° than both b and c, $\cos a$ is numerically greater than $\cos b \cos c$, and $\cos \alpha$ and $\cos a$ have like signs, so that α and a are of the same quadrant.

In the formula of Art. 95,

$$\tan\frac{\alpha+\beta}{2}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{\gamma}{2},$$

$\frac{1}{2}(a-b)$ and $\frac{1}{2}\gamma$ are less than 90°, and their functions are positive. Thus $\tan\frac{1}{2}(\alpha+\beta)$ and $\cos\frac{1}{2}(a-b)$ have the same sign, so that $\frac{1}{2}(\alpha+\beta)$ and $\frac{1}{2}(a-b)$ are of the same quadrant.

99. CASE I. **Given the three sides.** The solution is effected by the formulas of Art. 93, and the check by the law of sines, Art. 90. Note that the sum of the sides must be less than 360°, while the sum of the angles must lie between 180° and 540°.

EXAMPLE. Given $a = 103°\ 12.5'$, $b = 96°\ 41.3'$, $c = 68°\ 21.2'$. Note that α and γ must be of the same quadrants as a and c, respectively, since these sides differ more from 90° than does b.

$$\begin{array}{llll}
a=103°\ 12.5' & s=134°\ 07.5' & & \\
b=\ \ 96°\ 41.3' & s-a=\ \ 30°\ 55.0' & & \\
c=\ \ 68°\ 21.2' & s-b=\ \ 37°\ 26.2' & & \\
 & s-c=\ \ 65°\ 46.3' & & \\
\text{colog sin } 0.14398 & & & \\
\text{log sin } 9.71079-10 & \text{log tan } 0.08849 & & \frac{1}{2}\alpha=50°\ 47.8' \\
\text{log sin } 9.78382-10 & \text{log tan } 0.01546 & & \frac{1}{2}\beta=46°\ 01.2' \\
\text{log sin } 9.95996-10 & \text{log tan } 9.83932-10 & & \frac{1}{2}\gamma=34°\ 38.1' \\
2\underline{)29.59855-30} & & & \alpha=101°\ 35.6' \\
\text{log tan } r\ 19.79928-20 & & & \beta=\ \ 92°\ 02.4' \\
 & & & \gamma=\ \ 69°\ 16.2'
\end{array}$$

$$\begin{array}{llll}
\text{colog sin sides} & 0.01165 & 0.00297 & 0.03176 \\
\text{log sin angles} & \underline{9.99105-10} & \underline{9.99973-10} & \underline{9.97093-10} \\
 & 0.00270 & 0.00270 & 0.00269
\end{array}$$

Find the angles of the following triangles and check the results:

1. $a = 73°\ 16'$, $b = 83°\ 25'$, $c = 120°\ 33'$.
2. $a = 65°\ 10'$, $b = 38°\ 20'$, $c = 46°\ 40'$.
3. $a = 98°\ 16.4'$, $b = 112°\ 31.3'$, $c = 105°\ 43.5'$.
4. $a = 16°\ 43.6'$, $b = 22°\ 34.4'$, $c = 28°\ 23.2'$.

100. CASE II. **Given the three angles.** The solution is effected by the formulas of Art. 94, and the check by the law of sines. The sum of the angles lies between 180° and 540°, the sum of the sides between 0° and 360°.

EXAMPLE. Given $\alpha = 135^\circ 18'$, $\beta = 80^\circ 24'$, $\gamma = 42^\circ 46'$. Note that $\alpha > 90^\circ$, $\gamma < 90^\circ$.

$$\begin{array}{ll}
 & S=129^\circ 14' \\
\alpha=135^\circ 18' & S-\alpha=-6^\circ 04' \\
\beta=\ \ 80^\circ 24' & S-\beta=\ 48^\circ 50' \\
\gamma=\ \ 42^\circ 46' & S-\gamma=\ 86^\circ 28'
\end{array}$$

$$\begin{array}{rll}
\text{colog cos } 0.19895 & & \\
\text{log cos } 9.99756-10 & \text{log tan } 0.59522 & \tfrac{1}{2}a=75^\circ 45.0' \\
\text{log cos } 9.81839-10 & \text{log tan } 0.41605 & \tfrac{1}{2}b=69^\circ 00.6' \\
\text{log cos } \underline{8.78979-10} & \text{log tan } 9.38745-10 & \tfrac{1}{2}c=13^\circ 42.8' \\
2\overline{)28.80469-30} & & a=151^\circ 30.0' \\
\text{log cot } R\ 19.40234-20 & & b=138^\circ 01.2' \\
 & & c=\ \ 27^\circ 25.6'
\end{array}$$

$$\begin{array}{rlll}
\text{colog sin angles} & 0.15280 & 0.00612 & 0.16812 \\
\text{log sin sides} & \underline{9.07866-10} & \underline{9.82534-10} & \underline{9.66333-10} \\
 & 9.83146-10 & 9.83146-10 & 9.83145-10
\end{array}$$

Find and check the sides of the following triangles :

1. $\alpha =\ 69^\circ 16'$, $\beta =\ 82^\circ 28'$, $\gamma = 116^\circ 36'$.
2. $\alpha = 130^\circ 12'$, $\beta = 110^\circ 22'$, $\gamma = 120^\circ 36'$.
3. $\alpha =\ 81^\circ 13.5'$, $\beta = 123^\circ 22.3'$, $\gamma = 138^\circ 16.4'$.
4. $\alpha = 141^\circ 07.2'$, $\beta =\ 82^\circ 03.4'$, $\gamma = 127^\circ 05.6'$.

101. CASE III. **Given two sides and the included angle.** The solution and check are effected by Napier's analogies, Art 95.

EXAMPLE. Given $b = 78^\circ 15'$, $c = 110^\circ 35'$, $\alpha = 84^\circ 24'$. Note that $\tfrac{1}{2}(\beta + \gamma) > 90^\circ$, $\beta < 90^\circ$, $\gamma > 90^\circ$.

$$\begin{array}{lll}
b=\ \ 78^\circ 15' & \tfrac{1}{2}(c-b)\ =16^\circ 10' & \\
c=110^\circ 35' & \tfrac{1}{2}(c+b)\ =94^\circ 25' & \\
\underline{\alpha=\ \ 84^\circ 24'\ \ } & \ \ \tfrac{1}{2}\alpha\ =42^\circ 12' & \text{log sin } 9.44474-10 \\
\beta=\ \ 68^\circ 43.4' & \tfrac{1}{2}(\gamma-\beta)=17^\circ 07.1' & \text{colog sin } 0.00129 \\
\gamma=102^\circ 57.6' & \tfrac{1}{2}(\gamma+\beta)=85^\circ 50.5' & \text{log cot } \underline{0.04252} \\
a=\ \ 88^\circ 58.6' & \ \ \tfrac{1}{2}a\ =44^\circ 29.3' & \text{log tan } 9.48853-10
\end{array}$$

log cos 9.98248 − 10	log tan 9.46224 − 10	
colog cos 1.11346		log tan 1.11217
log cot 0.04252		
	colog sin 0.53114	colog cos 0.01968
log tan 1.13846	log sin 9.99886 − 10	log cos 8.86040 − 10
	log tan 9.99224 − 10	log tan 9.99225 − 10

Find and check the missing parts of the following triangles:

1. $a = 85^\circ\ 24'$, $c = 71^\circ\ 36'$, $\beta = 120^\circ\ 12'$.
2. $a = 61^\circ\ 14'$, $b = 100^\circ\ 26'$, $\gamma = 82^\circ\ 38'$.
3. $b = 131^\circ\ 19.3'$, $c = 83^\circ\ 25.5'$, $\alpha = 140^\circ\ 32'$.
4. $a = 112^\circ\ 17.4'$, $c = 125^\circ\ 24.6'$, $\beta = 81^\circ\ 42'$.

102. CASE IV. **Given two angles and the included side.** As in Case III, the solution and check are effected by Napier's analogies, Art. 95.

EXAMPLE. Given $\alpha = 145^\circ\ 18'$, $\gamma = 113^\circ\ 12'$, $b = 76^\circ\ 14'$. Note that $\frac{1}{2}(a + c) > 90^\circ$, $a > 90^\circ$, $c > 90^\circ$.

$\alpha = 145^\circ\ 18'$	$\frac{1}{2}(\alpha - \gamma) = 16^\circ\ 03'$	log sin 9.44166 − 10
$\gamma = 113^\circ\ 12'$	$\frac{1}{2}(\alpha + \gamma) = 129^\circ\ 15'$	colog sin 0.11104
$b = 76^\circ\ 14'$	$\frac{1}{2}b = 38^\circ\ 07'$	log tan 9.89463 − 10
$a = 145^\circ\ 39.0'$	$\frac{1}{2}(a - c) = 15^\circ\ 38.9'$	log tan 9.44733 − 10
$c = 114^\circ\ 21.2'$	$\frac{1}{2}(a + c) = 130^\circ\ 00.1'$	
$\beta = 101^\circ\ 30.0'$	$\frac{1}{2}\beta = 50^\circ\ 45.0'$	

log cos 9.98273 − 10	log cot 0.54108	
colog cos 0.19880		log cot 9.91224 − 10
log tan 9.89463 − 10		
	log sin 9.43093 − 10	log cos 9.98359 − 10
log tan 0.07616	colog sin 0.11576	colog cos 0.19194
	log tan 0.08777	log tan 0.08777

Find and check the missing parts of the following triangles:

1. $\alpha = 89^\circ\ 26'$, $\beta = 136^\circ\ 42'$, $c = 75^\circ\ 38'$.
2. $\beta = 138^\circ\ 32'$, $\gamma = 171^\circ\ 22'$, $a = 33^\circ\ 12'$.
3. $\beta = 48^\circ\ 29.4'$, $\gamma = 86^\circ\ 35.6'$, $a = 129^\circ\ 13'$.
4. $\alpha = 73^\circ\ 24.6'$, $\gamma = 55^\circ\ 17.4'$, $b = 145^\circ\ 27'$.

103. CASE V. **Given two sides and the angle opposite one.** The angle opposite the other given side is found by the law of sines, Art. 90, and the remaining side and angle are found and checked by Napier's analogies, Art. 95.

Let the given parts be a, b, α. Sin β is found by the formula $\sin\beta = \dfrac{\sin b \sin \alpha}{\sin a}$. In case $\sin a < \sin b \sin \alpha$, the value found for $\sin\beta$ is greater than unity, so that no solution is possible. In case $\sin a > \sin b \sin \alpha$, there will be one or two solutions, according as one or both of the supplementary values of β corresponding to the value found for $\sin\beta$ satisfy the condition that the greater side is opposite the greater angle.

EXAMPLE 1. Given $b = 28°\,30'$, $c = 80°\,25'$, $\beta = 46°\,15'$.

$$\begin{array}{rl}
b = 28°\,30' & \text{colog sin } 0.32134 \\
c = 80°\,25' & \text{log sin } 9.99390 - 10 \\
\beta = \underline{46°\,15'} & \text{log sin } \underline{9.85876 - 10} \\
\gamma = & \text{log sin } 0.17400
\end{array}$$

The solution is impossible.

EXAMPLE 2. Given $a = 48°\,21'$, $c = 103°\,35'$, $\gamma = 129°\,15'$.

$$\begin{array}{rlr}
a = 48°\,21' & \text{log sin } 9.87345 - 10 & \frac{1}{2}(c-a) = 27°\,37' \\
c = 103°\,35' & \text{colog sin } 0.01232 & \frac{1}{2}(c+a) = 75°\,58' \\
\gamma = \underline{129°\,15'} & \text{log sin } \underline{9.88896 - 10} & \frac{1}{2}(\gamma-\alpha) = 46°\,21.5' \\
\alpha = 36°\,32' & \text{log sin } 9.77473 - 10 & \frac{1}{2}(\gamma+\alpha) = 82°\,53.5' \\
\beta = 48°\,59.8' & & \frac{1}{2}\beta = 24°\,29.9' \\
b = 71°\,18.6' & & \frac{1}{2}b = 35°\,39.3'
\end{array}$$

$$\begin{array}{rrr}
\text{log sin } 9.66610-10 & \text{log cos } 9.94747-10 & \text{log cot } 0.28137 \\
\text{colog sin } 0.01316 & \text{colog cos } 0.61531 & \\
\text{log cot } 9.97940-10 & & \text{log sin } 9.85954-10 \\
 & \text{log cot } \underline{9.09589-10} & \text{colog sin } \underline{0.00335} \\
\text{log tan } \overline{9.65866-10} & 9.65867-10 & \\
 & & \text{log cot } 0.14426 \\
 & \text{log cot } 9.39785-10 & \\
 & \text{log cos } 9.83894-10 & \\
 & \text{colog cos } 0.90747 & \\
 & \overline{0.14426} &
\end{array}$$

The supplement of α is greater than γ, while a is less than c. There is, therefore, only one solution.

Find and check the missing parts of the following triangles:

1. $b = 67^\circ 35'$, $c = 58^\circ 27'$, $\beta = 62^\circ 34'$.
2. $a = 73^\circ 42'$, $b = 69^\circ 31'$, $\alpha = 112^\circ 12'$.
3. $a = 105^\circ 21.5'$, $c = 82^\circ 16.5'$, $\gamma = 75^\circ 32.6'$.
4. $b = 132^\circ 25.4'$, $c = 141^\circ 31.2'$, $\beta = 85^\circ 24.6'$.

104. CASE VI. **Given two angles and the side opposite one.** The side opposite the other given angle is found by the law of sines, Art. 90, and the remaining side and angle are found and checked by Napier's analogies, Art. 95. As in Case V, there may be one, two, or no possible solutions.

EXAMPLE. Given $\alpha = 35^\circ 43'$, $\beta = 49^\circ 17'$, $a = 33^\circ 24'$.

$\alpha = 35^\circ 43'$	colog sin 0.23375	$\frac{1}{2}(\beta - \alpha) = 6^\circ 47'$
$\beta = 49^\circ 17'$	log sin 9.87964 − 10	$\frac{1}{2}(\beta + \alpha) = 42^\circ 30'$
$a = 33^\circ 24'$	log sin 9.74074 − 10	$\frac{1}{2}(b - a) = 6^\circ 06.6'$
$b = 45^\circ 37.2'$	log sin 9.85413 − 10	$\frac{1}{2}(b + a) = 39^\circ 30.6'$
$c = 62^\circ 57.4'$		$\frac{1}{2}c = 31^\circ 28.7'$
$\gamma = 109^\circ 10.6'$		$\frac{1}{2}\gamma = 54^\circ 35.3'$

log sin 9.07231 − 10	log cos 9.99695 − 10	log cot 0.92464
log sin 0.17032	colog cos 0.13237	
log cot 0.97043		log sin 9.02710 − 10
	log cot 0.08374	colog sin 0.19640
log cot 0.21306	0.21306	
		log tan 0.14814
	log cot 0.03795	
	log cos 9.99752 − 10	
	colog cos 0.11266	
	0.14813	

. . . .

	$\frac{1}{2}(\beta - \alpha) = 6^\circ 47'$	log sin 9.07231 − 10
	$\frac{1}{2}(\beta + \alpha) = 42^\circ 30'$	colog sin 0.17032
$b' = 134^\circ 22.8'$	$\frac{1}{2}(b' - a) = 50^\circ 29.4'$	log cot 9.91626 − 10
	$\frac{1}{2}(b' + a) = 83^\circ 53.4'$	
$c' = 163^\circ 35.6'$	$\frac{1}{2}c' = 81^\circ 47.8'$	log cot 9.15889 − 10
$\gamma' = 162^\circ 34.2'$	$\frac{1}{2}\gamma' = 81^\circ 17.1'$	

log cos 9.99695 − 10	log cot 0.92464	
colog cos 0.13237		log cot 0.03795
	log sin 9.88734 − 10	log cos 9.80360 − 10
log cot 9.02957 − 10	colog sin 0.00248	colog cos 0.97290
9.15889 − 10		
	log tan 0.81446	0.81446

Note that the logarithms of the functions of $\frac{1}{2}(b' - a)$ are the cologarithms of the co-functions of $\frac{1}{2}(b + a)$, etc.

Find and check the missing parts of the following triangles:

1. $\alpha = 74^\circ 21'$, $\beta = 80^\circ 24'$, $b = 69^\circ 40'$.

2. $\beta = 65^\circ 48'$, $\gamma = 87^\circ 36'$, $c = 125^\circ 30'$.

3. $\alpha = 126^\circ 20.4'$, $\gamma = 134^\circ.45.8'$, $a = 75^\circ 42.5'$.

4. $\alpha = 84^\circ 42.5'$, $\beta = 65^\circ 42.6'$, $b = 136^\circ 47.4'$.

EXERCISE XXVI

Find the missing parts of the following triangles:

	α	β	γ	a	b	c
1.				70° 04.3′	63° 21.5′	59° 16.4′
2.	38° 19.3′	48° 00.2′	121° 08.1′			
3.	27° 22.6′				118° 02.2′	120° 18.6′
4.	81° 38.3′	70° 09.6′	64° 46.5′			
5.				93° 27.6′	100° 04.4′	96° 14.8′
6.			80° 28′		104° 26′	81° 12′
7.	49°	52°				64° 26′
8.			131° 11.2′	77° 25.2′		128° 13.8′
9.				68° 34.2′	59° 21.3′	112° 16.5′
10.	85° 12.7′	123° 19.3′			68° 24.6′	
11.		142° 27.4′		78° 29′		110° 54′
12.			132° 35.8′	63° 37.2′		99° 28.6′
13.	43° 18.8′	86° 36.6′	173° 13.2′			
14.	81° 25.3′	124° 23.1′				142° 20.4′
15.	63° 43.8′	75° 18.4′		56° 38.2′		
16.		45° 30′		130°		150°
17.	113° 26.3′		53° 13.7′	68° 14.6′		
18.	33° 33.4′	66° 06.6′				88° 18.8′
19.	73° 12.4′	29° 19.8′	140° 07.6′			
20.			58° 16.7′	80° 21.4′		60° 32.6′
21.		82° 43′	123° 21′	160° 32′		
22.	65° 13.4′				130° 26.6′	150° 39.8′
23.		71° 35.5′	47° 16.3′		115° 15.8′	
24.				123° 45.6′	134° 56.7′	156° 18.9′

FORMULAS

GENERAL TRIGONOMETRY

$$\csc\alpha = \frac{1}{\sin\alpha}.$$

$$\sec\alpha = \frac{1}{\cos\alpha}.$$

$$\cot\alpha = \frac{1}{\tan\alpha}.$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}.$$

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha}.$$

$$\sin^2\alpha + \cos^2\alpha = 1.$$

$$\tan^2\alpha + 1 = \sec^2\alpha.$$

$$\cot^2 + 1 = \csc^2\alpha.$$

$$2\,\pi^R = 360°.$$

$F(2k\pi + \alpha) = F(\alpha)$, k an integer.

$F\left(k\cdot\frac{\pi}{2} \pm \alpha\right) = \pm F(\alpha)$, k an even integer.

$F\left(k\cdot\frac{\pi}{2} \pm \alpha\right) = \pm$ co-$F(\alpha)$, k an odd integer.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta.$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}.$$

$$\cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}.$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha.$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}.$$

$$\cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha}.$$

$$\sin\tfrac{1}{2}\alpha = \sqrt{\tfrac{1}{2}(1-\cos\alpha)}.$$

$$\cos\tfrac{1}{2}\alpha = \sqrt{\tfrac{1}{2}(1+\cos\alpha)}.$$

$$\tan\frac{1}{2}\alpha = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha}.$$

$$\cot\frac{1}{2}\alpha = \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} = \frac{1+\cos\alpha}{\sin\alpha}.$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right].$$

$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right].$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-b)\right].$$

$$\sin\alpha\sin\beta = -\tfrac{1}{2}\left[\cos(\alpha+\beta) - \cos(\alpha-\beta)\right].$$

$$\sin\alpha\cos\alpha = \tfrac{1}{2}\sin 2\alpha.$$

$$\cos^2\alpha = \tfrac{1}{2}(1+\cos 2\alpha).$$

$$\sin^2\alpha = \tfrac{1}{2}(1-\cos 2\alpha).$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.$$

RIGHT TRIANGLES

$$a^2 + b^2 = c^2.$$

$$\alpha + \beta = 90^\circ.$$

$$\frac{a}{c} = \sin\alpha = \cos\beta.$$

$$\frac{b}{c} = \cos \alpha = \sin \beta.$$

$$\frac{a}{b} = \tan \alpha = \cot \beta.$$

$$A = \tfrac{1}{2}\, ab = \tfrac{1}{2}\, bc \sin \alpha = \tfrac{1}{2}\, c^2 \sin \alpha \cos \alpha = \tfrac{1}{4} c^2 \sin 2\, \alpha.$$

OBLIQUE TRIANGLES

$$\alpha + \beta + \gamma = 180^\circ.$$

$$c = b \cos \alpha + a \cos \beta, \text{ etc.}$$

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

$$c^2 = a^2 + b^2 - 2\, ab \cos \gamma, \text{ etc.}$$

$$\tan \frac{\beta - \gamma}{2} = \frac{b - c}{b + c} \cot \frac{\alpha}{2}, \text{ etc.}$$

$$\tan \frac{\alpha}{2} = \frac{r}{s - a}, \text{ etc.}$$

$$s = \tfrac{1}{2}(a + b + c).$$

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}.$$

$$A = \tfrac{1}{2}\, bc \sin \alpha = rs = \sqrt{s(s - a)(s - b)(s - c)}.$$

RIGHT SPHERICAL TRIANGLES

$$\sin a = \sin c \sin \alpha.$$

$$\sin b = \sin c \sin \beta.$$

$$\tan b = \tan c \cos \alpha.$$

$$\tan a = \tan c \cos \beta.$$

$$\tan a = \sin b \tan \alpha.$$

$$\tan b = \sin a \tan \beta.$$

$$\cos c = \cos a \cos b.$$

$$\cos \beta = \cos b \sin \alpha.$$

$$\cos \alpha = \cos a \sin \beta.$$

$$\cos c = \cot \alpha \cot \beta.$$

OBLIQUE SPHERICAL TRIANGLES

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma, \text{ etc.}$$

$$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c, \text{ etc.}$$

$$\tan \frac{\alpha}{2} = \frac{\tan r}{\sin (s - a)}, \text{ etc.}$$

$$s = \tfrac{1}{2}(a + b + c).$$

$$\tan r = \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}.$$

$$\tan \frac{a}{2} = \frac{\cos (S - \alpha)}{\cot R}, \text{ etc.}$$

$$S = \tfrac{1}{2}(\alpha + \beta + \gamma).$$

$$\cot R = \sqrt{-\frac{\cos (S - \alpha) \cos (S - \beta) \cos (S - \gamma)}{\cos S}}.$$

$$\tan \tfrac{1}{2}(\beta - \gamma) = \frac{\sin \frac{1}{2}(b - c)}{\sin \frac{1}{2}(b + c)} \cot \tfrac{1}{2}\alpha, \text{ etc.}$$

$$\tan \tfrac{1}{2}(\beta + \gamma) = \frac{\cos \frac{1}{2}(b - c)}{\cos \frac{1}{2}(b + c)} \cot \tfrac{1}{2}\alpha, \text{ etc.}$$

$$\tan \tfrac{1}{2}(b - c) = \frac{\sin \frac{1}{2}(\beta - \gamma)}{\sin \frac{1}{2}(\beta + \gamma)} \tan \tfrac{1}{2} a, \text{ etc.}$$

$$\tan \tfrac{1}{2}(b + c) = \frac{\cos \frac{1}{2}(\beta - \gamma)}{\cos \frac{1}{2}(\beta - \gamma)} \tan \tfrac{1}{2} a, \text{ etc.}$$

ANALYTIC TRIGONOMETRY

$$\lim_{\theta \doteq 0} \left[\frac{\theta}{\sin \theta}\right] = 1.$$

$$\lim_{\theta \doteq 0} \left[\frac{\theta}{\tan \theta}\right] = 1.$$

$$(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) = \cos (\alpha + \beta) + i \sin (\alpha + \beta).$$

$$(\cos \alpha + i \sin \alpha)^n = \cos n\alpha + i \sin n\alpha.$$

$$\cos \alpha + i \sin \alpha = e^{i\alpha}.$$

$$\cos \alpha - i \sin \alpha = e^{-i\alpha}.$$

$$\cos \alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}.$$

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}.$$

$$\cosh\alpha + \sinh\alpha = e^{\alpha}.$$

$$\cosh\alpha - \sinh\alpha = e^{-\alpha}.$$

$$\cosh\alpha = \frac{e^{\alpha} + e^{-\alpha}}{2}.$$

$$\sinh\alpha = \frac{e^{\alpha} - e^{-\alpha}}{2}.$$

$$e^{x} = \lim_{n \doteq \infty}\left[\left(1 + \frac{x}{n}\right)^{n}\right] = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = 2.71\cdots.$$

$$\log(1 + y) = \frac{y}{1} - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots.$$

$$\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots.$$

$$\sin\alpha = \frac{\alpha}{1!} - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \cdots.$$

$$\tan\alpha = \frac{\alpha}{1} + \frac{\alpha^3}{3} + \frac{\alpha^5}{15} + \frac{17\,\alpha^7}{315} + \cdots.$$

$$\cosh\alpha = 1 + \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \frac{\alpha^6}{6!} + \cdots.$$

$$\sinh\alpha = \frac{\alpha}{1!} + \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \frac{\alpha^7}{7!} + \cdots.$$

CONSTANTS

$i = \sqrt{-1}$.

$\pi = 3.14159265 \cdots$.

$\pi^{R} = 180°$.

$1^{R} = 57.2957795° \cdots = 57°\,17'\,44.8'' \cdots$.

$e = 2.7182818285 \cdots$.

$$\text{Mod}_e\, 10 = \frac{1}{\log_e 10} = .4342944819 \cdots.$$

1 inch = 2.54001 ··· centimeters.
1 foot = .3048 ··· meters.
1 mile = 1.60935 ··· kilometers.
1 centimeter = .3937 ··· inches.
1 meter = 3.28083 feet = 1.09361 yards.
1 kilometer = .62137 miles.
$g = 32.086528 + .171293 \sin^2 \phi$ feet per second per second.
$= 9.779886 + .05221 \sin^2 \phi$ meters per second per second at sea level for latitude ϕ.

ANSWERS TO EXERCISES

(Answers are omitted in case their knowledge would detract from the value of the exercise.)

Exercise II

6. $(0,\ 0),\ (a,\ 0),\ (a,\ a),\ (0,\ a);\ \left(\frac{a}{2}\sqrt{2}, 0\right),\ \left(0, \frac{a}{2}\sqrt{2}\right),\ \left(-\frac{a}{2}\sqrt{2}, 0\right),\ \left(0, -\frac{a}{2}\sqrt{2}\right)$.

7. $(5, 0),\ (0, -5),\ (-4.33, -2.5),\ (-3.54, 3.54)$.

9. 5.6569; 7.6158.

11. Cross country distances, in miles: 5.099; 2.828; 2.236; 2.236; 6.325.

12. Distances saved, in yards: 773.2; 644.4; 1288.7; 128.9.

Exercise IV

9. $\frac{1681}{1519}$.

10. $\frac{3}{2}$.

11. $\frac{29}{20}$.

12. $\frac{89}{65}$.

13. $\cos\alpha$.

14. $2\csc\beta$.

15. $\sqrt{\frac{1-\sin\gamma}{1+\sin\gamma}}$.

16. $2(1+\tan^2\gamma)$.

Exercise V

9. $\frac{1}{2}$.

10. $\frac{1}{2}\sqrt{3}$.

11. $\frac{5}{2}$.

12. $\frac{1}{6}(3\sqrt{3}-2)$.

13. 60°.

14. 0° and 60°.

15. 60° and 90°.

16. 45°.

17. (*a*) 60°; (*b*) 19.05 ft.

18. 8.08; 16.17.

19. 452.39.

20. 60°.

Exercise VI

11. 0° and 60°. **12.** 30°. **13.** 0°, 30°, and 45°. **14.** 0°, 30°, and 45°.

Exercise VII

1. (*a*) 6.7, 6.7; (*b*) 8.23, 4.75.

2. (*a*) 8; (*b*) 13.86.

3. (*a*) 20, 34.64; (*b*) 28.28, 28.28.

4. 15.6; 9.

5. (*a*) 2598.16; (*b*) 1500.

6. 24 miles per hour, 30° east of north.

Article 18

1. $a = 40.32,\ b = 11.76$.

2. $a = 20.25,\ c = 33.75$.

3. $b = 151.5,\ c = 381.6$.

4. $a = 133.2,\ b = 149.2$.

Exercise VIII

(These results were obtained with four-place tables.)

1. $\beta = 64° 50'$, $a = 14.46$, $b = 30.77$.
2. $\beta = 37° 40'$, $a = 57.79$, $b = 44.61$.
3. $\alpha = 28° 45'$, $a = 116.88$, $b = 213.04$.
4. $\alpha = 11° 25'$, $a = 103.11$, $b = 510.68$.
5. $\beta = 68° 35'$, $b = 599.13$, $c = 643.66$.
6. $\beta = 17° 15'$, $b = 223.56$, $c = 753.93$.
7. $\alpha = 9° 30'$, $b = 7170.96$, $c = 7272.73$.
8. $\alpha = 72° 30'$, $a = 4757.40$, $c = 4988.36$.
9. $\alpha = 41° 49'$, $\beta = 48° 11'$, $b = 268.33$.
10. $\alpha = 32° 12'$, $\beta = 57° 48'$, $b = 605.03$.
11. $\alpha = 34° 13'$, $\beta = 55° 47'$, $a = 354.25$.
12. $\alpha = 53° 8'$, $\beta = 36° 52'$, $a = 1120$.
13. $\alpha = 36° 52'$, $\beta = 53° 8'$, $c = 1080$.
14. $\alpha = 44° 46'$, $\beta = 45° 14'$, $c = 845.07$.
15. $\alpha = 29° 11'$, $\beta = 60° 49'$, $c = 440.94$.
16. $\alpha = 59° 41'$, $\beta = 30° 19'$, $c = 2445.55$.
17. $\alpha = 29° 29'$, $\beta = 60° 31'$, $b = 168.00$, $c = 193.00$.
18. $\alpha = 41° 04'$, $\beta = 48° 56'$, $a = 230.00$, $c = 350.13$.
19. $\beta = 15° 40'$, $a = 93.47$, $b = 26.21$, $c = 97.08$.
20. $\alpha = 65° 10'$, $a = 60.35$, $b = 27.93$, $c = 66.50$.

21. 200.1 ft.
22. 1501.73 ft.
23. 2° 23′.
24. 4° 46′.
25. 1° 9′.
26. 33° 41′, 26° 34′, 45°.

27. .134 pitch, .2887 pitch, ½ pitch.

28. 19° 28′ inclination.
29. 8° 3′ inclination.
30. 0° 9′, 0° 17′, 1° 26′.
31. 26° 34′.
32. 5859.71 ft.
33. 32° 28′.
34. 21.73 ft. = 21 ft. 8¾ in.
35. 260.4 ft.
36. 0° 20′.

37. 5° 54′ (=.1029 radians). Note that .1029 = sin 5° 52.5′, an approximation. See Arts. 72 and 77 (3).

38. 2.468 miles.
39. 502.2 ft.

40. $A = (0, 0)$, $B = (240.9, 0)$, $C = (385.9, 274.8)$,
$D = (98.7, 814.6)$, $E = (162.8, 1043.0)$, $F = (649.1, 1248.8)$.

41. $\beta = 110°$, $b = 68.81$, $A = 828.81$; $\beta = 36°$, $a = 202.27$, $A = 12{,}641.88$; $\alpha = 75° 06'$, $\beta = 29° 48'$, $A = 30{,}441.6$; $\alpha = 63°$, $b = 326.88$, $A = 52{,}425.01$; $\alpha = 64° 17'$, $a = 553.12$, $A = 119{,}594.88$; $\alpha = 41° 1'$, $\beta = 97° 58'$, $A = 202{,}809.6$.

42. 51.76, 61.80, 68.40, 76.54, 1, 121.76, 141.42, 173.20.

43. $S = 2\,s \cos\frac{\alpha}{4}$.

44. $s = 2\,R \sin\frac{180°}{n} = 2\,r \tan\frac{180°}{n}$.

45. $C = 2\,\pi R$, $P = 2\,nR \sin\frac{180°}{n}$, $c = 2\,\pi R \cos\frac{180°}{n}$.

$C = 62.83$, $P_4 = 56.57$, $c_4 = 44.43$;
$P_8 = 61.23$, $c_8 = 58.05$;
$P_{16} = 62.43$, $c_{16} = 61.62$;
$P_{32} = 62.72$, $c_{32} = 62.53$.

46. $A_c = \pi R^2, \quad A_p = nR^2 \sin\frac{180°}{n}\cos\frac{180°}{n} = \frac{1}{2}nR^2 \sin\frac{360°}{n},$

$$A_i = \pi R^2 \cos^2\frac{180°}{n} = \frac{1}{2}\pi R^2\left(1 + \cos\frac{360°}{n}\right).$$

$A_c = 314.16, \quad A_{p4} = 200.00, \quad A_{i4} = 157.08;$
$A_{p8} = 282.84, \quad A_{i8} = 268.15;$
$A_{p16} = 306.16, \quad A_{i16} = 302.21;$
$A_{p32} = 312.16, \quad A_{i32} = 311.14.$

47. $C = 62.83, \quad P_6 = 60.00, \quad c_6 = 54.41;$
$P_{12} = 62.11, \quad c_{12} = 60.69;$
$P_{24} = 62.64, \quad c_{24} = 62.29;$
$P_{48} = 62.78, \quad c_{48} = 62.69.$

48. $A_c = 314.16, \quad A_{p6} = 259.80, \quad A_{i6} = 235.62;$
$A_{p12} = 300.00, \quad A_{i12} = 293.11;$
$A_{p24} = 310.56, \quad A_{i24} = 308.80;$
$A_{p48} = 313.20, \quad A_{i48} = 312.81.$

49. $X = -30$, $Y = -17.321$, $R = 34.641$, 30° south of west.

50. $X = 6r$, $Y = 0$, $R = 6r$, due east.

51. Distance from center $= r\cos\theta$. **52.** $x = -15$.

53. Component along plane $= g\sin\alpha$, component perpendicular to plane $= g\cos\alpha$.

54. 16, 27.71; 8.28, 30.91; 5.56, 31.51; 2.79, 31.88.

55. 50 pounds pressure, 141.42 pounds along ladder.

56. $X = 57.28$, $Y = 30.73$. **57.** $R = 18.44$, $\theta = 49° 24'$. **58.** $c = 11.99$.

Exercise IX

1. 4, 1.5, -2.5.
2. 4, $\frac{5}{3}$, $\frac{2}{3}$.
3. 1, 4, 16, 32, 64, $\frac{1}{16}$, $\frac{1}{64}$.
4. 1, 8, 16, $\frac{1}{4}$, $\frac{1}{64}$.
5. (*a*) .4724; (*b*) .01614.
6. (*a*) 28.16; (*b*) .01913; (*c*) 2.465.
7. 17.978. **8.** .76252. **9.** 524.9. **10.** $x = 2.29$.
11. $x = 1.79$. **12.** $4656.20. **13.** $4136.09. **14.** 5.2 %.
15. 11.6 years. **16.** 3.8 years.

Exercise X

1. 4.86024, 2.79187, 9.84198, 5.80872 – 10, 21.47712.
2. 4.96088, 1.15518, 11.50651, 5.89510 – 10, 24.30103.
3. 516.35, 4.0966 × 10^{12}, .016335.
4. 16361, 5.64325 × 10^{11}, .00013671.
5. 9,067,800,000.
6. 7.0048 × 10^{10} cm.
7. 88.594 cm.
8. 71.68 cm.
9. 13,231 × 10^{10}.
10. 2,754,100.
11. 9.63459 – 10, 9.52928 – 10, 0.01824.
12. 9.97454 – 10, 9.78340 – 10, 0.04197.
13. 78° 01.1′, 81° 43.7′, 76° 17.1′.
14. 25° 20.7′, 27° 32.6′, 35° 3.6′.
15. 13.861. **16.** .91186. **17.** 3.9968. **18.** .38875.
19. 1.3365 inches.
20. .1111 foot (= 1.3332 inches).
21. 1.7 % less than the true value.
22. .074765.
23. 6.711; 8.381.
24. – 11.85; – 61.38.

Exercise XII

1. $\beta = 27^\circ$, $a = 2302.3$, $b = 1173.1$.
2. $\alpha = 60^\circ\,37.6'$, $\beta = 29^\circ\,22.4'$, $b = 4238.9$.
3. $\beta = 14^\circ\,44.8'$, $b = 254.07$, $c = 998.12$.
4. $\alpha = 15^\circ\,39.6'$, $\beta = 74^\circ\,20.4'$, $b = 168.36$, $c = 174.85$.
5. $\alpha = 50^\circ\,13.1'$, $\beta = 39^\circ\,46.9'$, $c = 9.5378$.
6. $\beta = 71^\circ\,34.5'$, $a = 10.417$, $b = 31.271$.
7. $\alpha = 83^\circ\,38.4'$, $\beta = 6^\circ\,21.6'$, $b = 14.82$, $c = 133.79$.
8. $\alpha = 75^\circ\,33'$, $\beta = 14^\circ\,27'$, $c = 54.953$.
9. $\alpha = 64^\circ\,48.5'$, $\beta = 25^\circ\,11.5'$, $b = 31{,}037$.
10. $\beta = 60^\circ\,9.8'$, $a = 5.854$, $c = 11.766$.
11. $\beta = 64^\circ\,42.6'$, $a = 19.023$, $b = 40.264$, $c = 44.531$.
12. $\alpha = 28^\circ\,23.6'$, $\beta = 61^\circ\,36.4'$, $c = .00042$.
13. $\alpha = 26^\circ\,47.3'$, $a = 3.2196$, $b = 6.3769$.
14. $\alpha = 38^\circ\,23.3'$, $\beta = 51^\circ\,36.7'$, $a = .056677$.
15. $\alpha = 54^\circ\,43.6'$, $b = .44535$, $c = .77120$.
16. $\alpha = 54^\circ\,43.2'$, $a = 242.79$, $b = 343.16$, $c = 420.37$.
17. $\alpha = 55^\circ\,59.3'$, $\beta = 34^\circ\,0.7'$, $c = .0074192$.
18. $\alpha = 9^\circ\,47.5'$, $a = .89928$, $c = 5.2878$.
19. $\alpha = 63^\circ\,20.7'$, $\beta = 26^\circ\,39.3'$, $a = .014523$.
20. $\alpha = 64^\circ\,41.8'$, $a = 1563.4$, $b = 739.12$.
21. 8.2583 feet.
22. 18 feet 3.8 inches.
23. 141.42 square feet.
24. 60° 1.8′.
25. 1237.8 feet.
26. 16⅔ miles, 36° 52.2′ north of west.
29. $nR^2 \sin\dfrac{180^\circ}{n}\cos\dfrac{180^\circ}{n}$.
30. 2177.4.
31. 178.8 miles.
32. 0° 32′.
33. .078523 feet.
34. 14834 feet.
35. 425.64 feet.
36. 142.4 feet.
37. 118.1 feet.
38. 554.06; 145.17.

Article 47

1. $\alpha = 33^\circ\,19.9'$, $a = 438.23$, $c = 788.58$.
2. $\alpha = 65^\circ\,49.8'$, $a = 122.13$, $b = 835.60$.
3. $\beta = 15^\circ\,57.0'$, $b = 5.442$, $c = 17.865$.
4. $\beta = 1^\circ\,02.0'$, $a = 9.368$, $b = .18134$.

Article 48

3. $\alpha = 57^\circ\,59.9'$, $\gamma = 23^\circ\,36.6'$, $c = 29.526$.
4. $\beta = 13^\circ\,55.6'$, $\gamma = 35^\circ\,30.7'$, $b = 135.96$.
5. $\begin{cases} \alpha = 104^\circ\,31.3',\ \beta = 40^\circ\,2.9',\ a = 5889.9; \\ \alpha = 4^\circ\,37.1',\ \beta' = 139^\circ\,57.1',\ a' = 489.8. \end{cases}$
6. $\begin{cases} \alpha = 94^\circ\,17.9',\ \gamma = 47^\circ\,13.3',\ a = 207{,}810; \\ \alpha' = 47^\circ\,4.6',\ \gamma' = 132^\circ\,46.7',\ a' = 152{,}600. \end{cases}$

Article 49

1. $\beta = 23^\circ\,42.8'$, $\gamma = 40^\circ\,45.2'$, $a = 450.35$.
2. $\alpha = 23^\circ\,31.8'$, $\gamma = 19^\circ\,7.2'$, $b = 818.54$.
3. $\alpha = 33^\circ\,17.5'$, $\gamma = 63^\circ\,13.1'$, $b = .11496$.
4. $\beta = 66^\circ\,27.0'$, $\gamma = 45^\circ\,11.2'$, $a = .005202$.

Article 50

1. $\alpha = 51^\circ\,57.2'$, $\beta = 66^\circ\,49.4'$, $\gamma = 61^\circ\,13.4'$; $A = 1.9181 \times 10^7$.
2. $\alpha = 82^\circ\,16.4'$, $\beta = 42^\circ\,51.8'$, $\gamma = 54^\circ\,51.8'$; $A = 4.4175 \times 10^7$.
3. $\alpha = 56^\circ\,28.8'$, $\beta = 34^\circ\,45.4'$, $\gamma = 88^\circ\,45.8'$.
4. $\alpha = 45^\circ\,53.4'$, $\beta = 72^\circ\,33.2'$, $\gamma = 61^\circ\,33.4'$.
5. Impossible. Why?
6. $\beta = 136^\circ\,39.8'$.
7. Impossible. Why?
8. $\gamma = 8^\circ\,58.3'$.

Exercise XIV

1. $\beta = 74^\circ\,0.3'$, $\gamma = 43^\circ\,24.7'$, $a = 76{,}568$.
2. $\alpha = 52^\circ\,56.6'$, $\beta = 79^\circ\,47.8'$, $\gamma = 47^\circ\,15.6'$.
3. $\gamma = 86^\circ\,10.3'$, $b = 8.4172$, $c = 9.0436$.
4. $\alpha = 43^\circ\,29.3'$, $\gamma = 80^\circ\,35.3'$, $b = .30470$.
5. Impossible. Why?
6. $\alpha = 52^\circ\,11.2'$, $\gamma = 27^\circ\,38.8'$, $a = 49{,}921$.
7. $\alpha = 34^\circ\,32.1'$, $\beta = 51^\circ\,41.8'$, $\gamma = 93^\circ\,46.1'$.
8. $\gamma = 69^\circ\,28.5'$, $a = 67{,}439$, $c = 72.037$.
9. $\alpha = 23^\circ\,34.1'$, $\beta = 35^\circ\,35.7'$, $c = 6.0804$.
10. $\alpha = 15^\circ\,35.2'$, $\gamma = 126^\circ\,7.6'$, $c = 66.113$.
11. $\beta = 13^\circ\,11.7'$, $\gamma = 16^\circ\,24.1'$, $a = .082764$.
12. $\left\{\begin{array}{l}\beta = 32^\circ\,8.5',\ \gamma = 89^\circ\,45',\ b = 34.993; \\ \beta' = 31^\circ\,38.5',\ \gamma' = 90^\circ\,15',\ b' = 36.210.\end{array}\right.$
13. $\alpha = 34^\circ\,11.5'$, $a = 382.48$, $c = 641.52$.
14. $\alpha = 28^\circ\,57.0'$, $\beta = 104^\circ\,28.6'$, $\gamma = 46^\circ\,34.4'$.
15. $\alpha = 21^\circ\,13.9'$, $\gamma = 32^\circ\,19.7'$, $b = .0048578$.
16. $\alpha = 162^\circ\,18.9'$, $\gamma = 7^\circ\,08.3'$, $a = 61.896$.
17. $\alpha = 33^\circ\,33.1'$, $\beta = 50^\circ\,42.0'$, $\gamma = 95^\circ\,44.9'$.
18. $\alpha = 75^\circ\,0.2'$, $a = 8355.2$, $b = 6470.6$.
19. $\alpha = 45^\circ\,29.5'$, $\beta = 14^\circ\,15.5'$; $b = 2146.7$.
20. $\alpha = 49^\circ\,36.8'$, $\beta = 40^\circ\,23.2'$, $c = 952.67$.
21. $\alpha = 151^\circ\,56.6'$, $\beta = 4^\circ\,30.4'$, $\gamma = 23^\circ\,38.0'$, $b = 416.45$.
22. $\alpha = 80^\circ\,0.0'$, $\beta = 54^\circ\,45.2'$, $\gamma = 45^\circ\,14.8'$, $a = 124.81$.
23. $\beta = 90^\circ\,50'$, $\gamma = 16^\circ\,0'$, $a = 720.81$, $c = 207.58$.
24. $\alpha = 95^\circ\,26.6'$, $\gamma = 27^\circ\,8.4'$, $a = 125.81$, $b = 106.49$.
25. 2.1815×10^9; 5.0105; $1{,}742{,}040{,}000$.
26. 2.567×10^9; $.038051$; 7270.3.
27. Case III.

28. $\begin{cases} \alpha = 15^\circ 45', \quad \beta = 29^\circ 15', \quad c = 52.1; \\ \alpha = 21^\circ 52.6', \beta = 42^\circ 7.4', c = 723.6; \\ \alpha = 91^\circ 54.7', \beta = 53^\circ 5.3', \quad c = 43.042; \\ \alpha = 47^\circ 59.5', \beta = 72^\circ 0.5', c = 291.38. \end{cases}$

29. $\begin{cases} A = 156^\circ 55.6', B = 145^\circ 57.2', C = 57^\circ 7.2'; \\ A = 145^\circ 13.6', B = 121^\circ 11.4', C = 93^\circ 35.0'. \end{cases}$

30. $\begin{cases} A = \ \ 63^\circ 25.9', B = 141^\circ 34.1', c = 328.4; \\ A = 122^\circ 25.4', C = 137^\circ 34.6', c = 575.41. \end{cases}$

31. $\begin{cases} A = 75^\circ, b = 878.48, c = 621.17; \\ A = 113^\circ 58.6', B = 106^\circ 2.2', C = 139^\circ 59.2'. \end{cases}$

32. $a = b = 48.5$.

33. $A = 151^\circ 2.7', B = 133^\circ 25.9', C = 75^\circ 31.4'$.

34. $d^2 = a^2 + b^2 + c^2 - 2\,ab \cos \widehat{ab} - 2\,bc \cos \widehat{bc} + 2\,ac \cos (\widehat{ab} + \widehat{bc})$; $d = 12.98$

35. $\widehat{bc} = 84^\circ 03.5'$, $\widehat{cd} = 75^\circ 53.0'$, $\widehat{da} = 82^\circ 03.5'$, $ab - cd = 109.28$, $bc - da = 114.47$.

36. $c = 1001.1$, $d = 568.6$. **37.** $\alpha = 36^\circ$, $s = 15.217$.

38. $V = 6\,\pi \arccos \frac{6-h}{6} - 30\,(6-h)\sqrt{12\,h - h^2}$: $V_1 = 135.0$, $V_2 = 371.7$,
$V_3 = 663.3$, $V_4 = 990.0$, $V_5 = 1338.0$, $V_6 = 1696.4$, $V_7 = 2054.8$, $V_8 = 2402.8$,
$V_9 = 2729.5$, $V_{10} = 3021.1$, $V_{11} = 3257.8$, $V_{12} = 3392.8$.

39. $b = 483.4$. **40.** $a = 3221.5$. **41.** $b = 1286$.

42. Distance = 31.63, total height = 20.97.

43. Distance = 24.24, height = 5.08.

44. $AD = 738.2$, $DB = 150.6$.

45. $AC = 1075.1$, $BC = 679.5$.

46. $AD = 1460$, $DC = 678$, angle BXC at left = $17^\circ 27.5'$.

47. Distance = 135.74, height = 36.602.

48. $D = 57^\circ 40'$, $CD = 196.73$, $BD = 233.55$.

49. $AD = 603.94$, $AC = 693.12$, $BC = 838.82$, $BE = 595.76$, $AB = 867.48$, angle CXB at left = $4^\circ 27.7'$.

50. $AC = 730.17$, $AD = 737.37$, $BE = 805.40$, $BF = 715.52$, $AB = 841.67$.

51. $39^\circ 54'$.

Exercise XV

1. 45°, 60°, 150°, 112° 30′, 171° 53′ 14.4″, 42° 58′ 18.6″.

2. $\frac{\pi^R}{6}, \frac{\pi^R}{12}, \frac{\pi^R}{4}, \frac{2\,\pi^R}{3}, \frac{4\,\pi^R}{3}, \frac{5\,\pi^R}{3}, \frac{5\,\pi^R}{2}$.

3. 31.416 cm., 62.832 cm., 125.664 cm., 47.124 cm.

4. $1^R, \frac{1^R}{2}, 2^R, \frac{\pi^R}{2}$.

5. Smaller sprocket: 4 revolutions per second, angular velocity = $8\,\pi$ radians per second, linear velocity of circumference = 201.06 inches per second; larger sprocket: $\frac{8}{3}$ revolutions per second, angular velocity = $\frac{16\,\pi}{3}$ radians per second, linear velocity of circumference = 201.06 inches per second. Speed of machine = 20.944 feet per second = 14.28 miles per hour.

6. Linear velocity of chain and of circumferences of both sprockets = 75.48 inches per second; angular velocity of larger sprocket = 15.09 radians per second, of smaller sprocket = 37.76 radians per second; smaller sprocket makes 5.1 revolutions per second.

13. $\frac{\pi}{4} < \alpha < \frac{5\pi}{4}$.

14. $0 < \alpha < \frac{\pi}{4}$, $\frac{\pi}{2} < \alpha < \frac{3\pi}{4}$, $\pi < \alpha < \frac{5\pi}{4}$, $\frac{3\pi}{2} < \alpha < \frac{7\pi}{4}$.

Exercise XVI

5. $\left(\frac{\pi}{6}, \frac{1}{2}\right)$, etc.

6. $\left(\frac{\pi}{4}, 1\right)$, etc.

7. $\left(\frac{\pi}{4}, 1\right)$, $\left(\frac{3\pi}{4}, -1\right)$, etc.

8. $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$, $\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$, etc.

Exercise XVII

1. $\frac{8}{5}$.
2. $-\frac{1+2\sqrt{2}}{3}$.
3. $\frac{5}{6}$.
4. $-\frac{58}{21}$.
5. 60°, 300°.
6. 120°, 300°.
7. 120°, 150°, 300°, 330°.
8. 15°, 75°, 135°, 195°, 255°, 315°.
9. 60°, 300°.
10. 30°, 150°, 210°, 330°.
11. 0°, 60°, 180°, 300°, 360°.
12. 0°, 150°, 180°, 210°, 360°.

Exercise XVIII

1. $-\sin 20°$.
2. $-\sin 10°$.
3. $\cot 14°$.
4. $-\cot 35°$.
5. $-\csc 30°$.
6. $\sec 40°$.
7. $\frac{1+\sqrt{3}}{2}$, $-\frac{2\sqrt{3}}{3}$.
8. $-\frac{1+\sqrt{3}}{2}$, $-\frac{2\sqrt{3}}{3}$.
9. 0, 0.
10. $-\frac{1+\sqrt{3}}{2}$, $\frac{2\sqrt{3}}{3}$.
11. $\frac{\sqrt{3}-1}{2}$, $-\frac{2\sqrt{3}}{3}$.
12. 0, 0.
19. $-\frac{1+\sqrt{3}}{4}$.
20. 0.
21. 0.
22. 0.
23. $-.35$.
24. $-\frac{1}{5}$.
25. $+\sqrt{1-a^2}$.
26. $-\frac{\sqrt{1-m^2}}{m}$.
27. $\sin\alpha$.
28. $-\sin\alpha$.
29. $\tan\alpha$.
30. $-\tan\alpha$.

Exercise XIX

5. $\alpha + \beta = \sin^{-1}\frac{36}{85} = \cos^{-1} -\frac{77}{85}$, II quadrant.
6. $\alpha + \beta = \arcsin -\frac{253}{325} = \arccos -\frac{204}{325}$, III quadrant.
7. $-\sin(\alpha+\beta)$.
8. $\cos(\alpha+\beta)$.
9. $\sin 2\alpha$.
10. $\cos 2\alpha$.
11. $\sin 2\theta$.
12. $\cos\theta$.

13. $105° = \arcsin\frac{\sqrt{2}+\sqrt{6}}{4} = \arccos\frac{\sqrt{2}-\sqrt{6}}{4}$;

$15° = 105° - 90° = \arcsin\frac{\sqrt{6}-\sqrt{2}}{4} = \arccos\frac{\sqrt{6}+\sqrt{2}}{4}$.

14. $75° = \arcsin\frac{\sqrt{6}+\sqrt{2}}{4} = \arccos\frac{\sqrt{6}-\sqrt{2}}{4}$;

$$15° = 90° - 75° = \arcsin\frac{\sqrt{6}-\sqrt{2}}{4} = \arccos\frac{\sqrt{6}+\sqrt{2}}{4}.$$

Exercise XX

11. $\tan 15° = 2 - \sqrt{3}$, $\cot 15° = 2 + \sqrt{3}$.

15. $\sin(\alpha+\beta+\gamma) = \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma$.

16. $\cos(\alpha+\beta+\gamma) = \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta\cos\gamma - \sin\alpha\cos\beta\sin\gamma - \cos\alpha\sin\beta\sin\gamma$.

17. $\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\beta\tan\gamma - \tan\gamma\tan\alpha - \tan\alpha\tan\beta}$.

18. $\cot(\alpha+\beta+\gamma) = \frac{\cot\beta\cot\gamma + \cot\gamma\cot\alpha + \cot\alpha\cot\gamma}{\cot\alpha\cot\beta\cot\gamma - \cot\alpha - \cot\beta - \cot\gamma}$.

Note the symmetry in the last four formulas.

Exercise XXI

5. $2\alpha = \sin^{-1}\frac{720}{1680} = \cos^{-1}\frac{1519}{1681}$.

6. $2\alpha = \arcsin \pm\frac{120}{169} = \arccos -\frac{119}{169}$. Explain the signs.

7. $\sin\frac{1}{2}\alpha = \pm\frac{3}{5}$ and $\pm\frac{4}{5}$, $\cos\frac{1}{2} = \pm\frac{4}{5}$ and $\pm\frac{3}{5}$.

8. $\sin\frac{1}{2}\alpha = \pm\frac{7\sqrt{2}}{26}$ and $\pm\frac{17\sqrt{2}}{26}$, $\cos\frac{1}{2}\alpha = \pm\frac{17\sqrt{2}}{26}$ and $\pm\frac{7\sqrt{2}}{26}$.

9. $\sin(\alpha+2\beta) = \frac{253}{325}$ and $-\frac{323}{325}$, $\cos(\alpha+2\beta) = -\frac{36}{325}$ and $-\frac{204}{325}$.

10. $\sin(\alpha-2\beta) = \pm\frac{253}{325}$ and $\mp\frac{323}{325}$, $\cos(\alpha-2\beta) = \mp\frac{36}{325}$ and $\mp\frac{204}{325}$.

17. $\alpha = 30°, 45°, 60°, 210°, 225°, 240°$. **18.** $\alpha = 90°, 270°$, and $\frac{1}{2}\arccos\frac{4}{5}$.

19. $\alpha = 67°\,30', 157°\,30', 247°\,30', 337°\,30'$, and $\frac{1}{2}\arctan\frac{3}{4}$.

20. $\alpha = 90°, 270°$, and $\frac{1}{2}\cos^{-1}\frac{5}{5}$. **23.** $2x$. **24.** 1. **25.** 0. **26.** 1.

Exercise XXII

1. $\frac{1}{2}[\sin 8\alpha + \sin 2\alpha]$.

2. $\frac{1}{2}[\sin 10\alpha - \sin 2\alpha]$.

3. $\frac{1}{2}[\cos 4\alpha - \cos 10\alpha]$.

4. $\frac{1}{2}[\cos 7\alpha + \cos 3\alpha]$.

5. $\frac{1}{4}[\cos\alpha - \cos 3\alpha]$.

6. $\frac{1}{8}[2\sin 2\alpha + \sin 4\alpha]$.

7. $\frac{1}{8}[3 + 4\cos 2\alpha + \cos 4\alpha]$.

8. $\frac{1}{8}[3 - 4\cos 2\alpha + \cos 4\alpha]$.

9. $\frac{1}{64}[3\sin 2\alpha - \sin 6\alpha]$.

10. $\frac{1}{16}[1 - \cos 4\alpha]$.

15. $k\cdot\frac{\pi}{4}$; $[k = 0, 1, 2, 3, 4]$.

16. $(2k+1)\frac{\pi}{4}$; $[k = 0, 1, 2, 3]$.

17. $(2k+1)\frac{\pi}{12}$; $[k = 0, 1, 2, \cdots 11]$ and $(3k+1)\frac{\pi}{6}$; $[k = 0, 1, 2, 3]$.

18. $(2k+1)\frac{\pi}{30}$; $[k = 0, 1, 2, \cdots 29]$ and $(2k+1)\frac{\pi}{10}$, $[k = 0, 1, 2, \cdots 9]$.

19. $k\cdot\frac{\pi}{4}$; $[k = 0, 1, 2, \cdots 7]$. **20.** $k\cdot\frac{\pi}{4}$; $[k = 0, 1, 2, \cdots 7]$.

Exercise XXIII

9. $(2k+1)\frac{\pi}{6}$; $[k - 0, 1, 2, \cdots]$.

10. $\frac{k\pi}{3}$. **11.** $\frac{k\pi}{4}$.

12. π and $(2k+1)\frac{\pi}{8}$.

13. $\frac{\cot^3\alpha - 3\cot\alpha}{3\cot^2\alpha - 1}$.

14. $\frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$.

15. $4\sin\alpha\cos^3\alpha - 4\sin^3\alpha\cos\alpha$.

16. $8\cos^4\alpha - 8\cos^2\alpha - 3$.

17. 0°, 15°, 105°, 180°, 255°, 345°. (See Exercise XIX, examples 13 and 14.)

18. $\alpha = 60°, 90°, 120°, 270°$, and $\arcsin - \frac{1}{2\sqrt{3}}$. **19.** $\frac{k\pi}{2}$. **20.** $\frac{k\pi}{2}$.

Article 72

1. $-\sin\theta$. **2.** $\sec\theta\tan\theta$. **3.** $-\csc\theta\cot\theta$. **4.** $\sec^2\theta$. **5.** $-\csc^2\theta$.

Article 73

5. $\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$; ± 1; $1, \frac{1 \pm \sqrt{-3}}{2}$; $\pm 1, \pm i$; $\pm 1, \frac{\pm 1 \pm \sqrt{-3}}{2}$.

6. $\cos\frac{(2k+1)\pi}{n} + i\sin\frac{(2k+1)\pi}{n}$; $\pm i$; $-1, \frac{1 \pm \sqrt{-3}}{2}$;
$\frac{\pm 1 \pm i}{2}$; $\pm i, \frac{\pm\sqrt{3} \pm i}{2}$.

Article 74

6. Products: $15 + 6i$; $-5 + 3i$; $6 + 12i$; $-8 - 20i$; $12 - 18i$; $4 + 22i$; $4 + 22i$.

7. Quotients: $5 - 3i$; $3 - 2i$. **8.** Results: $4 + 12i$; 1; -8; 1.

9. Roots: $4 - 3i$; $3 + 2i$; $-46 + 9i$; $\pm 2i$.

10. $\sqrt[n]{a}\left[\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right]$; $[k = 0, 1, 2, \cdots (n-1)]$.

11. ± 1; $\pm i$; $1, \frac{-1 \pm \sqrt{-3}}{2}$; $2, -1 \pm \sqrt{-3}$.

PART II

(Answers to examples in spherical trigonometry.)

Exercise XXIV

1. $\alpha = 40° 30.3'$, $\beta = 81° 13.5'$, $b = 76° 24.9'$.

2. $\beta = 33° 50.2'$, $a = 56° 47.0'$, $c = 61° 27.5'$.

3. $\alpha = 16° 06.2'$, $a = 12° 49.1'$, $b = 52° 00.6'$.

4. $\begin{cases} \beta = 28° 14.8',\ b = 25° 51.8',\ c = 67° 11.0'; \\ \beta' = 151° 45.2',\ b' = 154° 08.2',\ c' = 112° 49.0\,. \end{cases}$

5. $a = 21^\circ\, 06.6'$, $b = 169^\circ\, 46.0'$, $c = 156^\circ\, 38.6'$.
6. $\alpha = 49^\circ\, 34.6'$, $\beta = 97^\circ\, 43.4'$, $c = 96^\circ\, 38.0'$.
7. $\alpha = 84^\circ\, 47.1'$, $b = 127^\circ\, 53.7'$, $c = 94^\circ\, 02.8'$.
8. $\alpha = 92^\circ\, 31.0'$, $\beta = 163^\circ\, 01.4'$, $a = 98^\circ\, 38.8'$.
9. $\alpha = 49^\circ\, 36.7'$, $a = 40^\circ\, 23.3'$, $b = 46^\circ\, 21.8'$.
10. $\beta = 161^\circ\, 21.5'$, $a = 46^\circ\, 04.5'$, $b = 166^\circ\, 20.6'$.
11. $\alpha = 64^\circ\, 07.4'$, $b = 115^\circ\, 26.8'$, $c = 101^\circ\, 43.1'$.
12. $\alpha = 167^\circ\, 58.5'$, $\beta = 83^\circ\, 39.0'$, $a = 169^\circ\, 46.0'$.
13. $\begin{cases} \alpha = 13^\circ\, 59.4',\ a = 10^\circ\, 37.9',\ c = 130^\circ\, 15.3'. \\ \alpha' = 166^\circ\, 00.6',\ a' = 169^\circ\, 22.1',\ c' = 49^\circ\, 44.7'. \end{cases}$
14. $\beta = 157^\circ\, 26.1'$, $a = 48^\circ\, 28.1'$, $b = 162^\circ\, 43.2'$.
15. $\alpha = 142^\circ\, 48.0'$, $\beta = 82^\circ\, 57.6'$, $c = 99^\circ\, 25.2'$.
16. Impossible.

Exercise XXV

1. $\alpha = \beta = \gamma = 90^\circ$.
2. $\alpha = 158^\circ\, 54.3'$, $\gamma = 116^\circ\, 18.0'$, $b = 79^\circ\, 00.6'$.
3. $\gamma = 109^\circ\, 36.3'$, $a = 113^\circ\, 39.3'$, $b = 127^\circ\, 26.0'$.
4. $\alpha = \beta = 57^\circ\, 01.9'$, $c = 94^\circ\, 48.8'$.
5. $\alpha = 103^\circ\, 08.8'$, $a = 102^\circ\, 35.8'$, $b = 106^\circ\, 30.3'$.
6. $\alpha = 138^\circ\, 18.4'$, $\beta = 130^\circ\, 23.9'$, $b = 119^\circ\, 30.7'$.
7. $\alpha = \gamma = 113^\circ\, 48.0'$, $\beta = 96^\circ\, 17.7'$.
8. $\alpha = 67^\circ\, 55.3'$, $\beta = 52^\circ\, 00.7'$, $\gamma = 103^\circ\, 22.6'$.
9. $\gamma = 107^\circ\, 41.9'$, $a = 103^\circ\, 44.4'$, $b = 141^\circ\, 11.3'$.
10. $a = 68^\circ\, 50.2'$, $b = c = 107^\circ\, 12.6'$.
11. $\alpha = 58^\circ\, 03.2'$, $a = 45^\circ\, 05.8'$.
12. $\alpha = \beta = \gamma = 63^\circ\, 16.7'$.
13. $a = b = c = 113^\circ\, 42.8'$.
14. Impossible. (Why?)

Exercise XXVI

1. $\alpha = 81^\circ\, 38.2'$, $\beta = 70^\circ\, 09.7'$, $\gamma = 64^\circ\, 46.6'$.
2. $a = 46^\circ\, 24.8'$, $b = 60^\circ\, 14.6'$, $c = 89^\circ\, 00.4'$.
3. $\beta = 91^\circ\, 26.5'$, $\gamma = 102^\circ\, 06.1'$, $a = 23^\circ\, 57.2'$.
4. $a = 101^\circ\, 08.0'$, $b = 93^\circ\, 53.2'$, $c = 89^\circ\, 14.0'$.
5. $\alpha = 94^\circ\, 39.2'$, $\beta = 100^\circ\, 32.4'$, $\gamma = 96^\circ\, 58.6'$.
6. $\alpha = 85^\circ\, 57.6'$, $\beta = 104^\circ\, 55.0'$, $a = 88^\circ\, 19.6'$. (Why only one solution?)
7. $\gamma = 98^\circ\, 28.0'$, $a = 43^\circ\, 29.8'$, $b = 45^\circ\, 56.8'$.
8. $\begin{cases} \alpha = 69^\circ\, 14.2',\ \beta = 72^\circ\, 28.4',\ b = 84^\circ\, 29.6'; \\ \alpha' = 110^\circ\, 45.8',\ \beta' = 135^\circ\, 28.4',\ b' = 132^\circ\, 57.0'. \end{cases}$
9. $\alpha = 45^\circ\, 26.6'$, $\beta = 41^\circ\, 11.6'$, $\gamma = 134^\circ\, 54.0'$.
10. No solution. (Why?)
11. $\alpha = 99^\circ\, 53.0'$, $\gamma = 111^\circ\, 05.4'$, $b = 142^\circ\, 41.6'$.
12. $\alpha = 41^\circ\, 57.6'$, $\beta = 34^\circ\, 12.8'$, $b = 48^\circ\, 53.0'$.

13. $a = 79^\circ\,51.0'$, $b = 97^\circ\,00.8'$, $c = 136^\circ\,07.4'$.

14. $\gamma = 124^\circ\,10.6'$, $a = 46^\circ\,54.1'$, $b = 142^\circ\,26.5'$.

15. $\begin{cases} \gamma = 74^\circ\,10.0',\ b = 64^\circ\,16.8',\ c = 63^\circ\,38.4'; \\ \gamma' = 157^\circ\,19.8',\ b' = 115^\circ\,43.2',\ c' = 178^\circ\,29.4'. \end{cases}$

16. $\alpha = 76^\circ\,19.2'$, $\gamma = 140^\circ\,38.4'$, $c = 34^\circ\,53.0'$.

17. $\beta = 27^\circ\,05.2'$, $b = 27^\circ\,26.8'$, $c = 54^\circ\,11.0'$.

18. $\gamma = 108^\circ\,49.2'$, $a = 35^\circ\,42.9'$, $b = 74^\circ\,55.1'$.

19. $a = 147^\circ\,25.0'$, $b = 18^\circ\,14.4'$, $c = 154^\circ\,06.8'$.

20. $\begin{cases} \alpha = 74^\circ\,23.5',\ \beta = 104^\circ\,25.6',\ b = 97^\circ\,33.2'; \\ \alpha' = 105^\circ\,36.5',\ \beta' = 45^\circ\,14.2',\ b' = 46^\circ\,36.8'. \end{cases}$

21. $\alpha = 135^\circ\,21.6'$, $b = 28^\circ\,03.8'$, $c = 156^\circ\,39.6'$.

22. $\beta = 93^\circ\,18.3'$, $\gamma = 140^\circ\,00.9'$, $a = 43^\circ\,48.0'$.

23. $\alpha = 139^\circ\,47.4'$, $a = 142^\circ\,01.4'$, $c = 44^\circ\,26.3'$.

24. No solution. (Why?)

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